TYPES 2023

Differentiation as a monad

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What's your favorite monad ?

What's your favorite monad ?

A monad over a type A:

- It *encapsulate* a certain kind of values: $u_A : A \to M(A)$.
- It allows *computation* on these values: $\mu_A : M(M(A)) \to M(A)$

Examples:

- Partiality: $M : A \mapsto A + \perp$, $u_A : a \mapsto a$
- Non-determinism: $M : A \mapsto \mathcal{P}(A)$, $u_A : a \mapsto \{a\}$
- Effect: $M : A \mapsto (S \to (A \times S))$, $u_A : a \mapsto (s \mapsto (a, s))$

The continuation monad

$$\begin{aligned} u_A : A &\Rightarrow ((A \Rightarrow B) \Rightarrow B) \\ a &\mapsto \lambda k.ka \end{aligned}$$

The continuation monad, twisted

Linear arrow $\multimap$: using exactly once its argument

$$\begin{aligned} u_A : A &\Rightarrow ((A \Rightarrow B) \multimap B) \\ a &\mapsto \lambda k.ka \end{aligned}$$

The continuation monad

Linear arrow $\multimap$: using exactly once its argument

$$u_A : A \multimap ((A \Rightarrow B) \multimap B)$$
$$a \mapsto \lambda k.ka$$

Making k, a non-linear map, linear

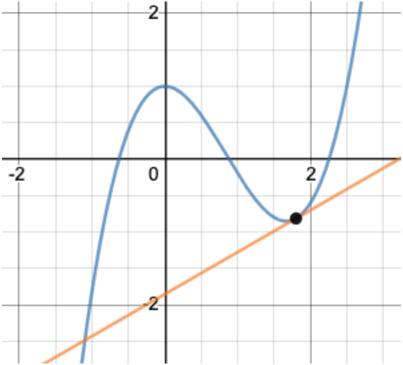
The continuation monad

Linear arrow $\multimap$: using exactly once its argument

$$u_A : A \multimap ((A \Rightarrow B) \multimap B)$$
$$a \mapsto \lambda k. D_0(k) a$$

Making k, a non-linear map, linear: differentiation

What's differentiation ?



The differential of a function at a point is its *best linear approximation* at that point.

From linearity to quantitative models

Functions	**Programs**
Power series	Resources consumption or Probabilistic sums
$f = \sum_n f_n$	$p(x) = \sum p_n$

From linearity to quantitative models

Functions	**Programs**
Power series $f = \sum_n f_n$	Resources consumption or Probabilistic sums $p(x) = \sum p_n$
f_n is n-linear	p_n consumes exactly n-times its resources.
f is *Taylor* $f = \sum_n \frac{1}{n!} D_0^{(n)} f$	Programs can be approximated $(M)S = \sum_n \frac{1}{n!} < M > S^{\otimes^n}$

- Experimentally, quantitative semantics is what gets you higher-order.
- It leads to new proof techniques on λ-calculus.
- A strong link with intersection types.

Simona Ronchi della Rocca's talk tomorrow!

Even when trying to avoid it, we stumble back on quantitative constructions [Dabrowski, K. 2018]

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Core intuition: Differentials are enough to compute

The quantitative monad

Theorem [K. Lemay 2023]

The following:

- $M : E \to \mathcal{C}^\infty(E, \mathbb{K})'$
- $u : v \mapsto (f \mapsto D_0(f)(v))$
- $\mu : \delta_\phi \mapsto \sum \frac{1}{n!} \phi^{*^n}$

is a monad in quantitative models of λ-calculus:

$$!u; \mu = id \qquad \Leftrightarrow \qquad f = \sum_n \frac{1}{n!} D_0^{(n)} f$$

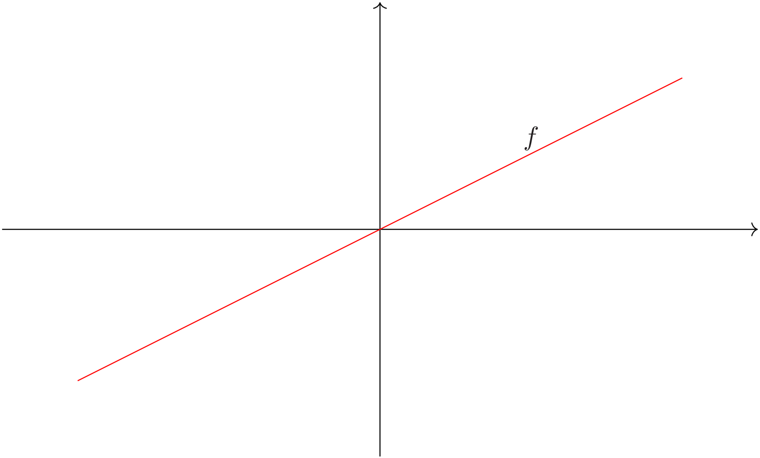
The monad laws:

$$u_M; \mu = id$$
$$M(u); \mu = id$$
$$\mu_M; \mu = M(\mu); \mu$$

From functional analysis to functional programming, and back

Surprise test

Is it a function ?



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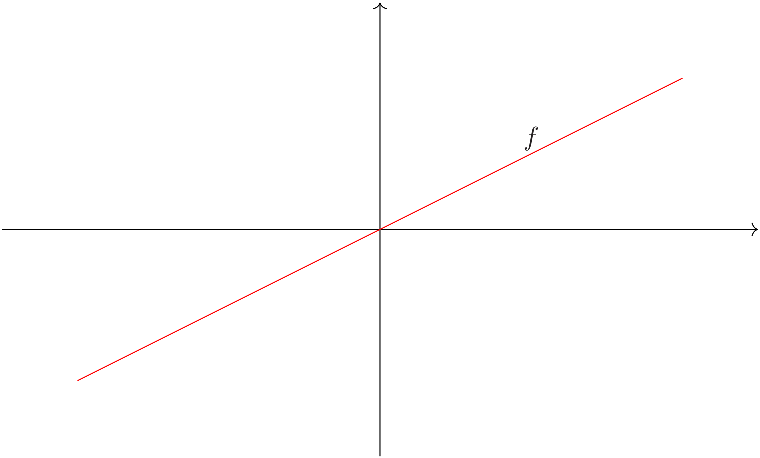

Yes, that's a linear function $f \in \mathscr{L}(\mathbb{R}, \mathbb{R})$

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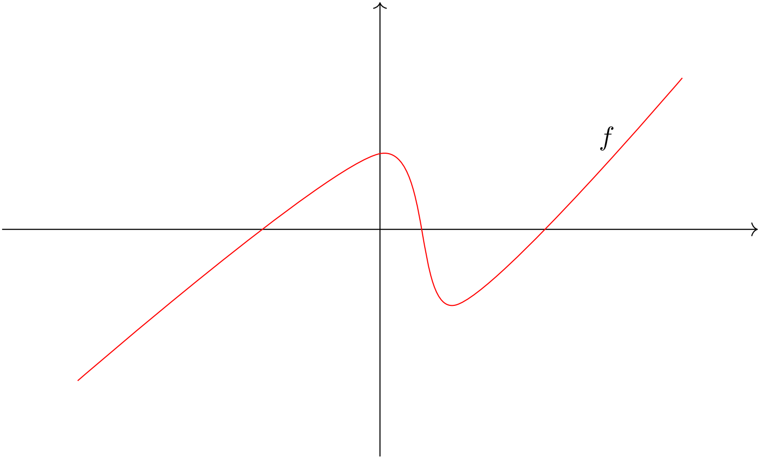

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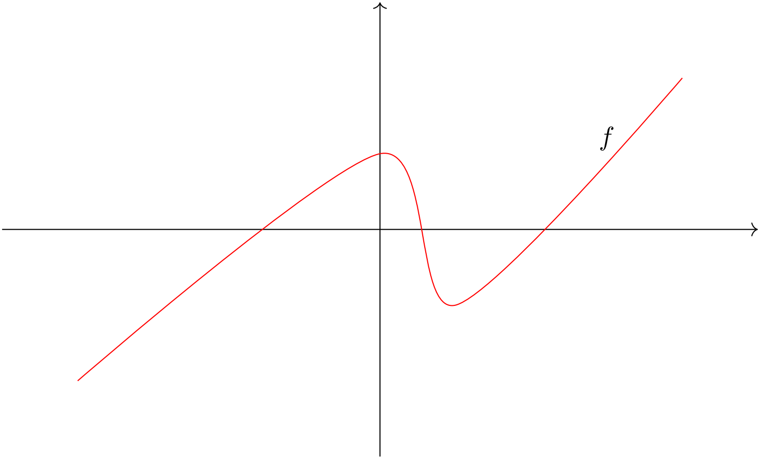

Yes, that's a smooth function $f \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$

Surprise test

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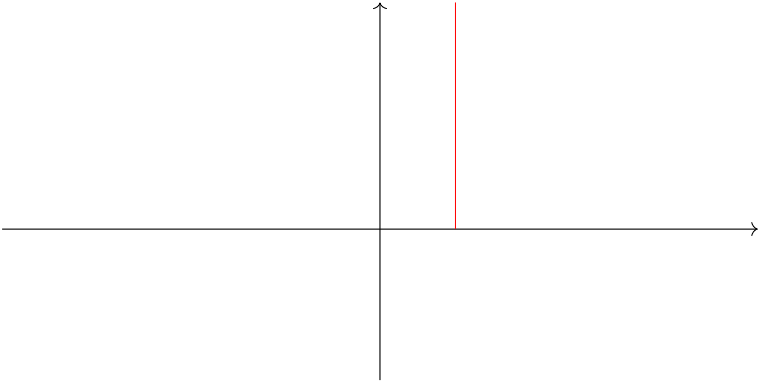

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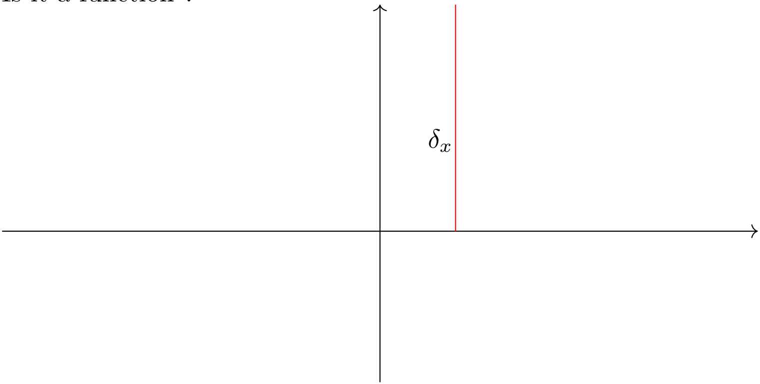


No, that's:

- ☐ A distribution
- ☐ A generalized function
- ■ That's the argument to a program.

Programs are interpreted as functions...

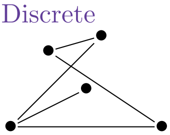
Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality

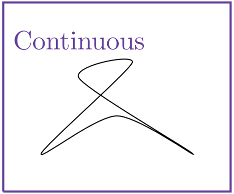
.. but *special* ones.

$$\text{Programs act on programs } f : \mathcal{C}(A, B) \to C$$

- (AxO) Domains A and spaces of functions $\mathcal{C}(A, B)$ are of the same kind.
- (AxF) Programs and function compute on several arguments:

$$f : A \times B \to C \equiv f : A \to \mathcal{C}(B, C)$$





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- Lattices
- Graphs
- Sequences

- Games
- Vector spaces
- ~~Normed spaces~~
- **Topological vector spaces**

Interpreting programs by smooth functions

$$p : A \Rightarrow B \qquad f \in \mathcal{C}^\infty(A, B)$$

Probabilistic Programming

$$p \xrightarrow{\alpha} x$$

Differentiable Programming

$$\mathrm{D}(p_1; p_2) = \mathrm{D}(p_1); \mathrm{D}(p_2)$$

- Correctness Properties $[\![\mathrm{D}(p)]\!] = \mathrm{D}([\![p]\!])$
- Completeness Properties $\forall f, \exists p, [\![p]\!] = f$
- **New** programming paradigms $p = \mathsf{d}(q)$
- **New** mathematical structures $\mathcal{C}^\infty(E, F)$

Convenient vector spaces
a first interpretation of Higher-Order Smooth Functions

$$(\mathrm{AxF}) \colon \mathcal{C}^\infty(A \times B, C) \simeq \mathcal{C}^\infty(A, \mathcal{C}^\infty(B, C))$$

Frölicher, Kriegl, Michor (1997)

Blute, Ehrhard, Tasson (2012)

Perspective

Programs	**Logic**	**Semantics**
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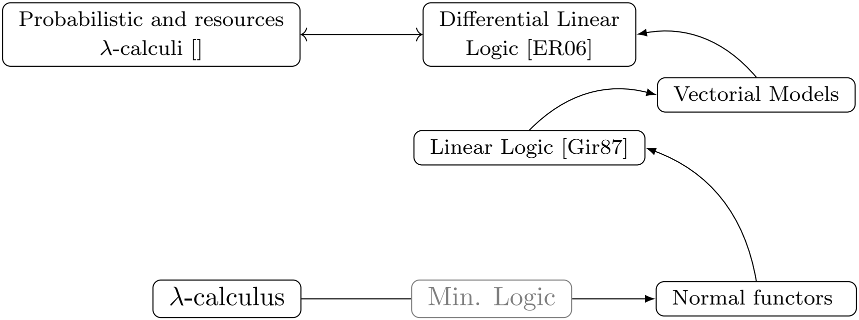

Interpreting programs by Linear Functions

$$[\![p]\!] \in \mathcal{L}(A, B)$$

(Ax0): If B is a complete or metrizable space, then so is $\mathcal{L}(A, B)$.

Trickier for A though

(AxF):

$$\mathcal{L}(A \otimes B, C) \simeq \mathcal{L}(A, \mathcal{L}(B, C))$$

- Always true algebraically.

Interpreting programs by Linear Functions

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(AxF):

$$\mathcal{L}_{\mathrm{B}}(A \otimes_{\mathrm{B}} B, C) \simeq \mathcal{L}_{\mathrm{B}}(A, \mathcal{L}_{\mathrm{B}}(B, C))$$

- Always true algebraically.
- Topologically, it depends on the set $\mathrm{B} \subset \mathcal{P}(A)$ of bounded sets on which uniform convergence must be enforced.
- MANY topological tensor products: $\otimes_\beta$, $\otimes_\sigma$, $\otimes_\mu$, $\otimes_\varepsilon$.
- MANY duals: $E'_{\mathrm{B}} := \mathcal{L}_{\mathrm{B}}(E, \mathbb{R})$

We are missing an important criteria

Not Not ... Who's there ?

Not Not ... Who's there ?

$$((A \Rightarrow \perp) \Rightarrow \perp) \simeq A$$

$\mathcal{C}^\infty(\mathcal{C}^\infty(A, \mathbb{K}), \mathbb{K}) \simeq A$

***No one**: not a chance for A smooth enough*

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$$((A \Rightarrow \bot) \Rightarrow \bot) \simeq A$$

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***No one**: not a chance for A smooth enough*

$$((A \multimap \bot) \multimap \bot) \simeq A$$

$\mathscr{L}(\mathscr{L}(A, \mathbb{K}), \mathbb{K}) \simeq A$

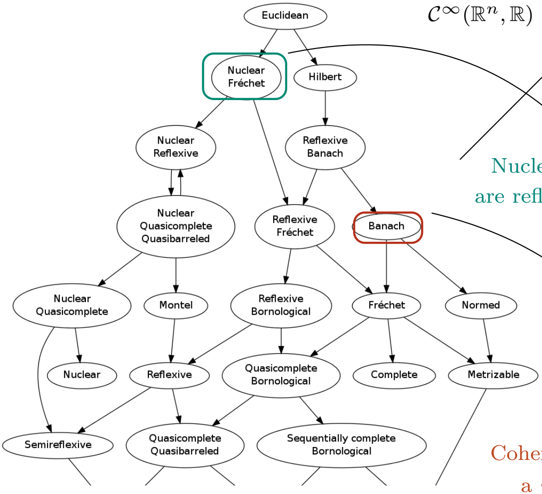
A lot of people!: Reflexive topological vector spaces.

We have plenty of examples!

- Finite dimensional vector spaces
- Hilbert spaces
- Spaces on which an orthogonality relation can be defined ...

In general, reflexive spaces enjoy `poor stability properties`.
$\times$ higher-order, $\times$ tensor product.

Interpreting types by reflexive topological vector spaces



$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

Nuclear Fréchet spaces are reflexive and complete

Coherent Banach spaces, Girard 2004, a *norm* is too restrictive

Let us take the other way around, through Nuclear, Complete+Metrizable (=Fréchet) spaces.

Polarization as a solution to reflexivity

Semantics for polarized MLL : Melliès Chiralities

$$\begin{tikzcd}(\mathscr{P}^{op}, \otimes, 1) \arrow[rr, bend left, "(\_)^{\perp_L}"] \arrow[rr, phantom, "\perp"] & & (\mathscr{N}, \parr, \perp) \arrow[ll, bend left, "(\_)^{\perp_R}"]\end{tikzcd} \qquad \begin{tikzcd}\mathscr{P} \arrow[rr, bend left, "\uparrow"] \arrow[rr, phantom, "\perp"] & & \mathscr{N} \arrow[ll, bend left, "\downarrow"]\end{tikzcd}$$

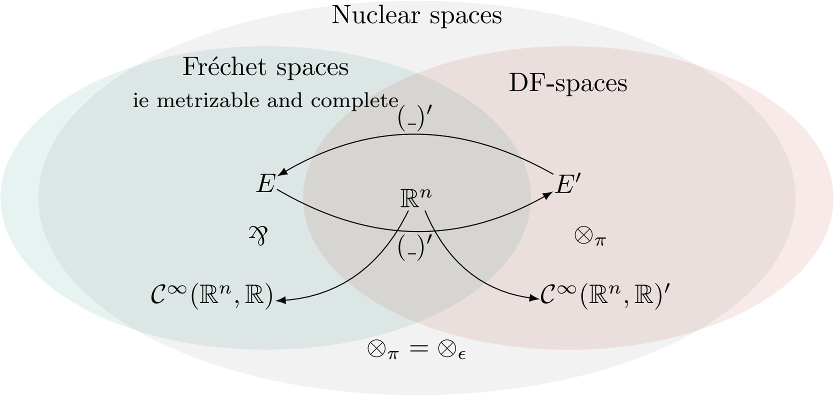
$$N^{\perp_R \perp_L} \simeq N$$

Replacing (AxF) with:

$$\mathscr{N}(\uparrow p \otimes n^{\perp_L}, m) \simeq \mathscr{N}(\uparrow p, n \parr m)$$

Interpreting formulas by *<u>two</u>* *categories of topological vector spaces, with a contravariant equivalence interpreting the involutive linear negation*

Polarization as a solution to reflexivity



Notation : $E' := \mathscr{L}(E, \mathbb{R})$

Grothendieck, Produits tensoriels topologiques et espaces nucléaires, 1958

Melliès, A micrological study of negation, APAL 2017

K. A Logical Account for Linear Partial Differential Equations, LICS 2018 .

Linear implications and reflexivity

Old and dusty mathematicians

Property: $E \simeq (E'_\beta)'_\beta \Leftrightarrow E$ barrelled and E weakly quasi complete.

Barrelled spaces (Bourbaki): there for Banach-Steinhauss theorem.

Theorem

- Barrelled and weak quasi-complete form a model of polarized calculus (Melliès' Chiralities).
- Banach-Steinhaus is exactly (**AxF**)!

$$\mathscr{N}(\uparrow p \otimes n^{\perp_L}, m) \simeq \mathscr{N}(\uparrow p, n \parr m)$$

Mixing Linear and Non-Linear Proofs: here comes the fun!

Not not ... Who's there ?

$$(A \Rightarrow \perp) \multimap \perp$$

$$\mathscr{L}(\mathcal{C}^\infty(A, \mathbb{R}), \mathbb{R}) = \mathcal{C}^\infty(A, \mathbb{R})'$$

Not not ... Who's there ?

$$(A \Rightarrow \bot) \multimap \bot$$

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Semantics	**Programs**
Distributions	Context
$\phi \in \mathcal{C}^\infty(A, \mathbb{R})'$	$C : (p : A \to \bot) \mapsto (\text{value} : \bot)$
e.g.: $\delta_x : f \mapsto f(x)$	$[\_](x) : p \to p[x]$

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e.g.: $\delta_x : f \mapsto f(x)$	$[\_](x) : p \to p[x]$
Reflexivity: $f(x) = \delta_x(f)$	$p(x) = < [\_](x) \vert p >$

Not not ... Who's there ?

$$(A \Rightarrow \bot) \multimap \bot$$

$$\mathscr{L}(\mathcal{C}^\infty(A,\mathbb{R}),\mathbb{R}) = \mathcal{C}^\infty(A,\mathbb{R})'$$

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e.g.: $\delta_x : f \mapsto f(x)$	$[\_](x) : p \to p[x]$
Reflexivity: $f(x) = \delta_x(f)$	$p(x) =< [\_](x) \mid p >$
Differentiation $D_0(f)(x) =< D_0(\_)(x) \mid f >$	

Laurent Schwartz, Théorie des distributions, 1950

Distributions: Linear Contexts for Non-Linear Programs

$$\mathcal{C}^\infty(E, F)'$$

- (AxO) for distributions:
  - $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is always Nuclear Fréchet and $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ is Nuclear DF.
  - If F is Fréchet, then $\mathcal{C}^\infty(\mathbb{R}^n, F)$ is Fréchet
  - Higher order: a bit of work.
- (AxF) for distributions:
  - For linear maps: $\mathscr{L}_\beta(\hat{E}, \mathscr{L}_\beta(F, G)) \simeq \mathscr{L}_\beta(\widehat{E \otimes_\beta F}, G)$ ✓
  - For smooth maps: $\mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G)) \simeq \mathcal{C}^\infty(E \times F, G)$ **?**
  - From one to another:

Schwartz' Kernel Theorem ✓

$$\mathcal{C}^\infty(E, \mathbb{K})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{K})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{K})'$$

A monoidal operation on distributions

$$(\phi \in \mathcal{C}^\infty(E,\mathbb{R})' \otimes \psi \in \mathcal{C}^\infty(E,\mathbb{R})') \mapsto ?$$

Convolution, the monoidal operation on distributions:

$$\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y)))$$

Different from $\phi + \psi : f \mapsto \phi(f) + \psi(g)$
Examples:

$$\delta_x * \delta_y = \delta_{x+y}$$
$$\delta_x * \mathrm{D}_0(\_)(v) = \mathrm{D}_x(\_)(v)$$
$$\mathrm{D}_0(\_)(v) * \mathrm{D}_0(\_)(v) = \mathrm{D}_0^{(2)}(\_)(v)$$

There is no "multiplication" extending from functions to distributions, this is our multiplication !

Quantitative semantics, another look

$$\forall x, \forall v, f(x) = \sum_n \frac{1}{n!} D_0^{(n)} f(x)$$

Quantitative semantics, another look

$$\forall f, \forall x, \forall v, < f | \delta_x >= \sum_n \frac{1}{n!} < f | D_0^{(n)}(\_)(x) >$$

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Quantitative semantics, another look

$$\forall x, \forall v, \delta_x = \underbrace{\sum_n \frac{1}{n!}}_{e} \overbrace{\mathrm{D}_0(\_)(x) * \cdots * \mathrm{D}_0(\_)(x)}^{\mathrm{D}_0(\_)(x)^{*^n}}$$

Quantitative semantics, another look

$$\forall x, \forall v, \delta_x = \sum_n \frac{1}{n!} \mathrm{D}_0(\_)(x) * \cdots * \mathrm{D}_0(\_)(x)$$

$$e^x = \sum_n \frac{1}{n!} x^n \qquad id = e^* \circ (\mathrm{D}_0(\_))$$

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A Quantitative Monad

- A functor $E \mapsto \mathcal{C}^\infty(E, \mathbb{R})'$ acting on a subcategory $\mathscr{L}$ of topological vector spaces and linear maps.
- Differentiation as a unit: $u : x \mapsto D_0(\_)(x)$
- The convolutional exponential as a multiplication: $\mu : \delta_\phi \mapsto \sum_n \frac{1}{n!} \phi^{*^n}$

Monad $\rightsquigarrow \forall f \in \mathscr{L}_!(A, B) \simeq \mathcal{C}^\infty(A, B)$, f is Taylor.

Examples: Relational model, Weighted Relational Model, Species, Nuclear Fréchet spaces

It was never about the quantitative semantics of λ-calculus.

Differential Linear Logic: from resources to distributions, from discrete to continuous settings

Exponential rules of (Differential) Linear Logic

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.

Exponential rules of (Differential) Linear Logic

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Resources calculi	**LINEAR LOGIC**	**Topological vector spaces**

Exponential connectives:

$$[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{K})'$$

$$[\![?B]\!] := \mathcal{C}^\infty([\![B]\!]', \mathbb{K})$$

Linear Logic, Jean-Yves Girard 1987

Differential Interaction Nets, Thomas Erhard and Laurent Regnier, 2006

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\, !A \multimap B$$

- Usual non-linear implication

A linear proof is in particular non-linear.

$A \vdash B$ is linear.

$!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma}\ \text{dereliction}$$

Slogan: ! in the hypotheses, speaking of resources.

Linear Logic

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Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\, !A \multimap B$$

- Usual non-linear implication
- Linear implication
- Exponential: Usually, the duplicable copies of A.

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Differential Linear Logic: co-structural rules

linear proof *non-linear proof*

$$\begin{tikzcd} \boxed{A \vdash B} \arrow[rr, phantom, "\text{Linear Logic}"] \arrow[rr, bend right, dashed] & & \boxed{!A \vdash B} \arrow[ll, bend right, dashed] \end{tikzcd}$$

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d},\ \text{dereliction}$$

linear $\hookrightarrow$ non-linear.

$$\frac{f : !A \vdash B}{\mathrm{D}_0(f) : A \vdash B}\ \bar{\mathsf{d}},\ \text{co-dereliction}$$

non-linear $\hookrightarrow$ linear

Differential Linear Logic: co-structural rules

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$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\,\bar{\mathsf{d}}$$

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non-linear $\hookrightarrow$ linear

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : !A}{\vdash \Gamma, !A}\,\bar{\mathsf{d}} \qquad \dfrac{\ell : A \vdash B}{\ell : !A \vdash B}\,\mathsf{d}\text{, dereliction}}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\rightsquigarrow$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,\text{cut}$$

Differential Linear Logic: co-structural rules

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Cut-elimination:

$$\dfrac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\,\bar{\mathsf{d}} \qquad \dfrac{\ell : A \vdash B}{\ell : !A \vdash B}\,\mathsf{d},\ \text{dereliction}}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\rightsquigarrow \qquad \dfrac{\vdash \Gamma, v : A \qquad \ell :\vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(v) = \ell(v)}\,\text{cut}$$

Dereliction and co-dereliction:

linear proof *non-linear proof*

$$\begin{tikzcd} \boxed{A \vdash B} \arrow[rr, bend right, dashed] & \text{DiLL} & \boxed{!A \vdash B} \arrow[ll, bend right, dashed] \end{tikzcd}$$

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\,\mathsf{d}$$

linear $\hookrightarrow$ non-linear.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\,\bar{\mathsf{d}}$$

non-linear $\hookrightarrow$ linear

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\,\bar{\mathsf{d}} \qquad \dfrac{\ell : A \vdash B}{\ell : !A \vdash B}\,\mathsf{d}, \text{ dereliction}}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\rightsquigarrow \qquad \frac{\vdash \Gamma, v : A \qquad \ell :\vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(v) = \ell(v)}\,\text{cut}$$

From resources to functions and distributions

(Co)-weakening

$$\frac{c :\vdash \Gamma}{cst_c : !A \vdash \Gamma}\, w$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w}$$

The constant function is non-linear

One can evaluate a function at 0

(Co)-contraction

$$\frac{x : !A, y : !A \vdash g(x, y) : \Gamma}{x : !A \vdash g(x, x) : \Gamma}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c}$$

The multiplication of scalar functions

Convolution of distributions

Symmetric cut-eliminations procedures

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{c}_A"] \arrow[dr, dash] & !A \otimes !A \arrow[d, "1_{!A} \otimes \mathsf{w}_A"] \\
!A \otimes !A \arrow[u, "1_{!A} \otimes \mathsf{w}_A"] & !A \arrow[l, "\mathsf{c}_A"]
\end{tikzcd}$$

The function cst_1 is neutral for scalar multiplication

$$\begin{tikzcd}
!A \arrow[d, "1_{!A} \otimes \overline{\mathsf{w}}_A"'] \arrow[dr, dash] & !A \otimes !A \arrow[l, "\overline{\mathsf{c}}_A"'] \\
!A \otimes !A \arrow[r, "\overline{\mathsf{c}}_A"'] & !A \arrow[u, "1_{!A} \otimes \overline{\mathsf{w}}_A"']
\end{tikzcd}$$

The dirac at 0 is neutral for the convolution

$$\begin{tikzcd}
A & \\
!A \arrow[u, "\mathsf{d}_A"] & !A \otimes !A \arrow[l, "\overline{\mathsf{c}}_A"] \arrow[ul, "\mathsf{w}_A \otimes \mathsf{d}_A + \mathsf{d}_A \otimes \mathsf{w}_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "\overline{\mathsf{d}}_A"'] \arrow[dr, "\overline{\mathsf{w}}_A \otimes \overline{\mathsf{d}}_A + \overline{\mathsf{d}}_A \otimes \mathsf{w}_A"] & \\
!A \arrow[r, "c_A"'] & !A \otimes !A
\end{tikzcd}$$

$\phi * \psi(\ell) = \phi(\ell)\psi(cst_1) + \psi(\ell)\phi(cst_1)$

$D_0(f \cdot g) = D_0(f) \cdot g(0) + D_0(g) \cdot f(0)$

Symmetric cut-elimination procedures

$$\overline{\mathsf{d}}; w = 0 \text{ and } \overline{\mathsf{w}}; d = 0$$

$$D_0(cst_1) = 0 \text{ and } \ell(0) = 0$$

$$\begin{tikzcd}
!A \otimes !A \arrow[d, "\overline{\mathsf{c}}_A"'] \arrow[dr, "\mathsf{w}_A \otimes \mathsf{w}_A"] & \\
!A \arrow[r, "\mathsf{w}_A"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !A & \\
!A \arrow[u, "\mathsf{c}_A"] & 1 \arrow[l, "\overline{\mathsf{w}}_A"] \arrow[ul, "\overline{\mathsf{w}}_A \otimes \overline{\mathsf{w}}_A"']
\end{tikzcd}$$

$$(\phi * \psi)(cst_1) = \phi(cst_1) \cdot \psi(cst_1) \qquad (f \cdot g)(0) = f(0) \cdot g(0)$$

$$\otimes = \cdot \text{ in } \mathbb{R}$$

Finitary differential Linear Logic

The first version by Erhard and Regnier in 2006:

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}}$$

It's a maths world.

Higher-Order

Higher-Order via promotion

Exponential rules of Linear Logic (Resources)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

Exponential rules added by Differential Linear Logic (Distributions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \quad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}}$$

The promotion rule $\mathsf{p} : !A \rightarrow !!A \quad \delta_x \mapsto \delta_{\delta_x}$**:**

- Makes $(!, \underset{\cdot}{,} \mathsf{p})$ a co-monad : $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- Is a co-monoidal operation on $!A$: $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- The cut-elimination between p and $\bar{\mathsf{d}}$ express the chain rule:

$$D_0(g \circ f) = D_{f(0)} g \circ D_0 f \qquad \bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$$

Codigging

Exponential rules of Linear Logic (Resources or functions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

Exponential rules added by Differential Linear Logic (Distributions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \quad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}} \quad \frac{?\Gamma \vdash x : A}{?\Gamma \vdash \_ : ?A}\, \bar{p}$$

Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- $\overline{\mathsf{d}}; \mathsf{p} = \overline{\mathsf{w}} \otimes \overline{\mathsf{d}}; \mathsf{p} \otimes \overline{\mathsf{d}}; \overline{\mathsf{c}}$

..

Co-digging ? $\overline{\mathsf{p}} : !!A \to !A$**:**

- $\overline{\mathsf{d}}; \overline{\mathsf{p}} = \mathsf{id}$
- $\overline{\mathsf{c}}; \overline{\mathsf{p}} = \overline{\mathsf{p}} \otimes \overline{\mathsf{p}}; \overline{\mathsf{c}}$
- $\overline{\mathsf{p}}; \mathsf{d} = \mathsf{c}; \overline{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$

Codigging

Exponential rules of Linear Logic (Resources or functions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p}$$

Exponential rules added by Differential Linear Logic (Distributions)

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- $\mathsf{p}; \mathsf{d} = \mathsf{id}.$
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- $\bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$

..

Co-digging ? $\bar{\mathsf{p}} : !!A \to !A$**:** $g : !A \Rightarrow !A$

- $\bar{\mathsf{d}}; \bar{\mathsf{p}} = \mathsf{id} \qquad \mathrm{D}_0(g) = id.$
- $\bar{\mathsf{c}}; \bar{\mathsf{p}} = \bar{\mathsf{p}} \otimes \bar{\mathsf{p}}; \bar{\mathsf{c}} \qquad g(x+y) = g(x) * g(y)$
- $\bar{\mathsf{p}}; \mathsf{d} = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$

The missing rule of Differential Linear Logic

Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p};\mathsf{d} = \mathsf{id}$.
- $\mathsf{p};\mathsf{c} = \mathsf{c};\mathsf{p} \otimes \mathsf{p}$
- $\bar{\mathsf{d}};\mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}};\mathsf{p} \otimes \bar{\mathsf{d}};\bar{\mathsf{c}}$

Co-digging ? $\bar{\mathsf{p}} : !!A \to !A$**:** $g : !A \Rightarrow !A$

- $\bar{\mathsf{d}};\bar{\mathsf{p}} = \mathsf{id}$ $\quad \mathrm{D}_0(g) = id$.
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- $\bar{\mathsf{p}};\mathsf{d} = \mathsf{c};\bar{\mathsf{p}} \otimes \mathsf{d};\mathsf{w} \otimes \mathsf{d}$?

Implicit definition of the exponential:

$$g = exp^* : \phi \mapsto \sum_n \frac{1}{n!}\phi^* \qquad \bar{\mathsf{p}} : \delta_\phi \mapsto \sum_n \frac{1}{n!}{\phi^*}^n$$

The missing rule of Differential Linear Logic

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The co-chain rule:

$$\bar{\mathsf{p}}(\delta_\phi)(\ell) = \sum_{n \geq 0} \frac{1}{n!}{\phi^*}^n(\ell) = \sum_{n \geq 1} \frac{n}{n!}\phi(l) \cdot {\phi^*}^{(n-1)}(cst_1) = \phi(\ell) \cdot \bar{\mathsf{p}}(\delta_\phi)(cst_1) = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$$

The missing rule of Differential Linear Logic

Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- $\bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$

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Implicit definition of the exponential:

$$g = exp^* : \phi \mapsto \sum_n \frac{1}{n!}\phi^* \qquad \bar{\mathsf{p}} : \delta_\phi \mapsto \sum_n \frac{1}{n!}\phi^{*^n}$$

The co-chain rule:

$$\bar{\mathsf{p}}(\delta_\phi)(\ell) = \sum_{n \geq 0} \frac{1}{n!}\phi^{*^n}(\ell) = \sum_{n \geq 1} \frac{n}{n!}\phi(l) \cdot \phi^{*^{(n-1)}}(cst_1) = \phi(\ell) \cdot \bar{\mathsf{p}}(\delta_\phi)(cst_1) = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$$

The monadic rules:

$$!\bar{\mathsf{d}}; \bar{\mathsf{p}} = id \qquad \forall v, \bar{\mathsf{p}}(\delta_{D_0(\_)(v)}) = \delta_v \qquad \forall v, \forall f, \sum_n \frac{1}{n!}\mathrm{D}_0^{(n)} f(v) = f(v)$$

A completely uniform logical and categorical structure

Exponential connectives:

$$[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{K})' \qquad [\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{K})$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p$$

Linear Logic [Girard1987]

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c}$$

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}} \qquad \boxed{\frac{?\Gamma \vdash x : A}{?\Gamma \vdash e^{x|\_} : ?A}\, \bar{p}}$$

Differential Linear Logic [Ehrhard&Regnier2006]

[Kerjean & Pacaud Lemay 2023]

A reason for this symmetry

Exponential connectives:

$$[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{K})' \qquad [\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{K})$$

Do you remember the Laplace transformation ?

A continuous version of a power series

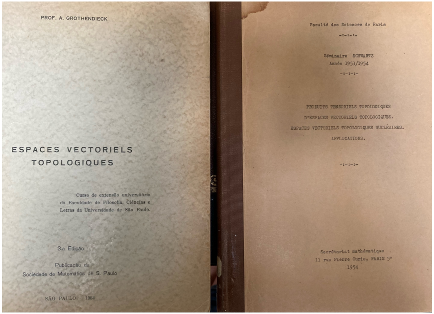
$$\mathscr{L} : \begin{cases} !E & \to ?E \\ \phi & \mapsto (\ell^{E'} \mapsto \phi(y^E \mapsto e^{<\ell|y>})) \end{cases}$$

$$\mathscr{L}(\overline{\mathsf{w}}, \overline{\mathsf{c}}, \overline{\mathsf{d}}, \overline{\mathsf{p}}) = \mathsf{w}, \mathsf{c}, \mathsf{d}, \mathsf{p}$$

- $M : E \to !E := [\![([\![E]\!] \Rightarrow \perp) \multimap \perp]\!] = \mathcal{C}^\infty([\![E]\!], \mathbb{K})'$
- $u : v \mapsto (f \mapsto D_0(f)(v))$
- $\mu : \delta_\phi \mapsto \sum \frac{1}{n!} \phi^{*^n}$

- $M : E \to !E := [\![([\![E]\!] \Rightarrow \bot) \multimap \bot]\!] = \mathcal{C}^\infty([\![E]\!], \mathbb{K})'$
- $u : v \mapsto (f \mapsto D_0(f)(v))$
- $\mu : \delta_\phi \mapsto \sum \frac{1}{n!} \phi^{*^n}$

The way I make things concrete



Existential questions

Does $\overline{\mathsf{p}} = e^*$ even exists ?

`Bad news: sums need to converge.`

At least at every point: $\left(\forall f, \forall \phi, \sum_n \frac{1}{n!} {\phi^*}^n(f)\right) \in \mathbb{R}$

- In discrete models of computations (e.g: relations over sets), sums are union, not an issue.
- In continuous models of computations, well...

Convolutional exponential VS exponential maps

(AxC) Proofs/Programs/Functions must compose

$$\overline{\mathsf{p}} : \delta_{\phi} \mapsto \sum \frac{1}{n!} \phi^{*^{n}}$$

Convolutional exponential VS exponential maps

(AxC) Proofs/Programs/Functions must compose

$$\overline{\mathsf{p}} : \delta_{\delta_x} \mapsto \sum \frac{1}{n!}\delta_{nx} = \left(f \mapsto \sum \frac{1}{n!} f(nx) \right)$$

Convolutional exponential VS exponential maps

(AxC) Proofs/Programs/Functions must compose

$$\overline{\mathsf{p}}:$$

$$\overline{\mathsf{p}}(\delta_{\delta_x})(x \mapsto e^x) = \sum \frac{1}{n!} e^{nx} = e^{e^x} \qquad \checkmark$$

$$\overline{\mathsf{p}}(\delta_{\delta_x})(x \mapsto e^{e^x}) = \sum \frac{1}{n!} e^{e^{nx}} \qquad \times$$

This example is due to T. Ehrhard

- p and $\overline{\mathsf{p}}$ do not mix well.
- We need to *quantify* over the exponential growth of the functions $\overline{\mathsf{p}}$ is applied to.

Grading Exponentials

$A \multimap B$

Linear programs/proofs/functions using exactly once their **resource**

$!A \multimap B$

Usual programs/proofs/functions

$$!_{\mathbf{n}} A \multimap B$$

- n-linear functions.
- Programs/proofs using exactly n-times their resource.
- Quantitative semantics: $!A = \sum !_n A$.

Exponentials indexed by semi-rings S

$$\forall \mathsf{s} \in \mathsf{S}, !_{\mathsf{s}} A \multimap B$$

Jean-Yves Girard, Andre Scedrov, Philip J. Scott, 1992

Martin Hoffman, Ugo Dal Lago,2009

Applications in implicit complexity and differential privacy.

From resources to differential equations

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p$$

$$\frac{}{\vdash \delta_0 : !A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c}$$

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}} \qquad \frac{?\Gamma \vdash x : A}{?\Gamma \vdash e^{x|\_} : ?A}\, \bar{p}$$

From resources to differential equations

Grading LL: a story of resources, again

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?_0 A}\, w \qquad \frac{\vdash \Gamma, f : ?_{\mathbf{x}} A, g : ?_{\mathbf{y}} A}{\vdash \Gamma, f.g : ?_{\mathbf{x+y}} A}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?_{\mathbf{1}} A}\, \mathsf{d} \qquad \frac{!\Gamma_y \vdash x : A}{!\Gamma_{z \times y} \vdash \delta_x : !_z A}\, p$$

$$\frac{}{\vdash \delta_0 : !A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c}$$

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}} \qquad \frac{?\Gamma \vdash x : A}{?\Gamma \vdash e^{x|\_} : ?A}\, \bar{p}$$

From resources to differential equations

Grading LL: a story of resources, again

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c}$$

$$\frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p$$

Grading DiLL: well, not so clear

$$\frac{}{\vdash \delta_0 : !_? A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !_? A \qquad \vdash \Delta, \psi : !_? A}{\vdash \Gamma, \Delta, \psi * \phi : !_? A}\, \bar{c}$$

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !_? A}\, \bar{\mathsf{d}} \qquad \frac{?_? \Gamma \vdash x : A}{?_? \Gamma \vdash e^{x|\_} : ?_? A}\, \bar{p}$$

Joining Quantitative, Graded and Differential Linear Logic

(AxC) Proofs/Programs/Functions must compose

$$\mu : \delta_{\delta_x} \mapsto \sum \frac{1}{n!}\delta_{nx} = \left(f \mapsto \sum \frac{1}{n!} f(nx) \right)$$

$$\mu(\delta_{\delta_x})(x \mapsto e^x) = \sum \tfrac{1}{n!} e^{nx} = e^{e^x n} \ \checkmark$$
$$\mu(\delta_{\delta_x})(x \mapsto e^{e^x}) = \sum \tfrac{1}{n!} e^{e^{nx}} \ \times$$

The quantitative monad does not apply to all functions, but only to those whose convergence is exponentially bound, according to some Young function θ.

$$?_{\theta,m}(F) := \{ f : F' \to \mathbb{C}, \forall m, \exists K, \forall z, |f(z)| \leq K e^{\theta(m||z||)} \}.$$

$$\overline{\mathsf{p}} : !_{\theta_1} (!_{\theta_2}(E)) \to !_{\left(\theta_1{}^* e^{\theta_2{}^*}\right)^*}(E)$$

Gannoun, Hachaichi, Ouerdiane, et Rezgui, Un théorème de dualité entre espaes de fonctions holomorphes à croissance exponentielles, 1999

Les fonctions parlent aux fonctions

$$!_{\theta,m}(F) := \{f : F \to \mathbb{C}, \forall m, \exists K, \forall z, |f(z)| \leq K e^{\theta(m||z||)}\}'.$$

`issue`: $||z||$: F needs to be normed.

- One single norm is too restrictive: we want to quantify over the convergence of the derivative of each function.
- The power of Fréchet spaces comes from their descriptions as a countable limit of Banach spaces:
$$F' := \lim_p N'_p$$

$$!_\theta F := \lim_{m,p} (?_{m,\theta} F_p)'$$

Les fonctions parlent aux fonctions

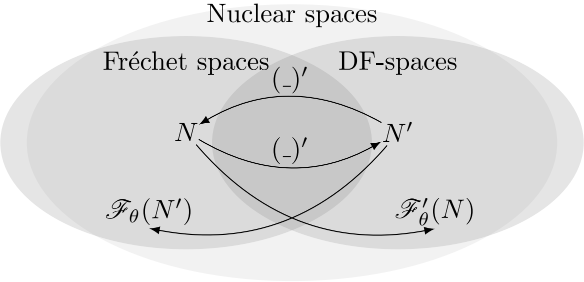
We have a quantitative and graded monad of Nuclear Dual of Fréchet spaces.

$$!_\theta F_p : \{f : F_p \to \mathbb{C}, \forall m, \exists K, \forall z, |f(z)| \leq K e^{\theta(m||z||)}\}'$$

$$!_\theta F := (\lim_{m,p} (?_{m,\theta} F_p)')'$$

The space of Young functions is a semi-ring with a new duality operation $(\_)^\star$.

$$\Theta : \{\theta, +, (\_ \times e^{\_}), (\_)^\star\}$$



Recap

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Types	Formulas	Objects
Execution	Cut-elimination	Equality

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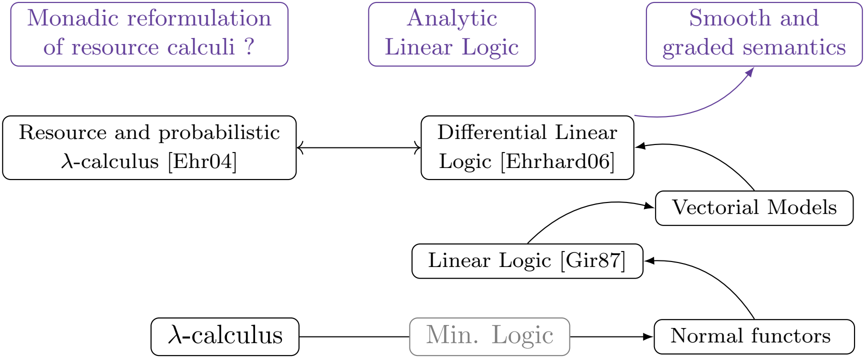

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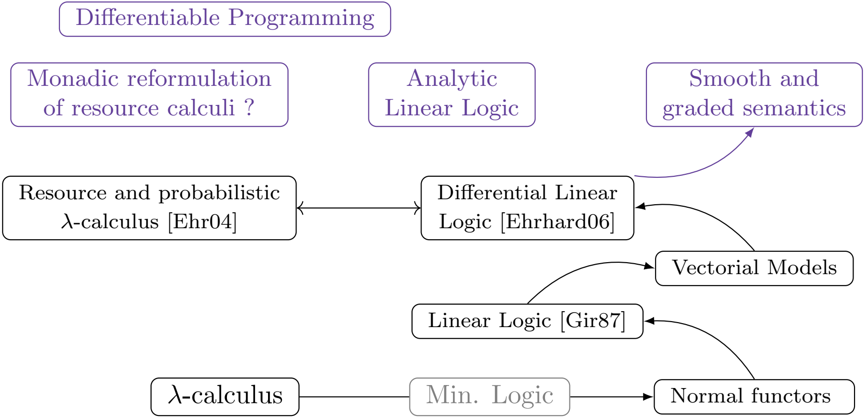

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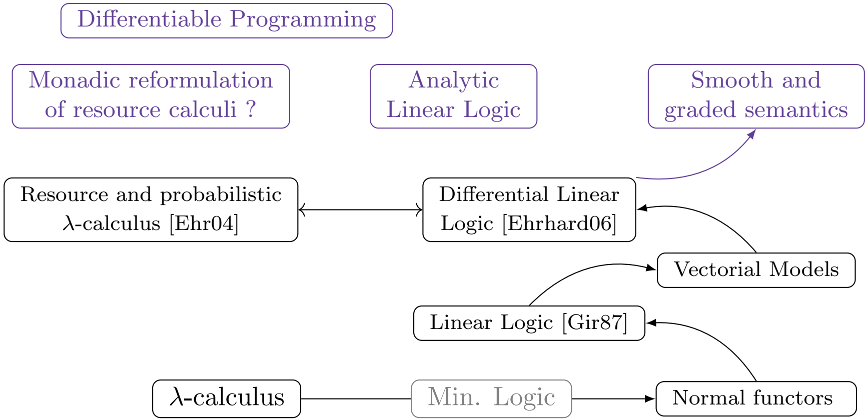

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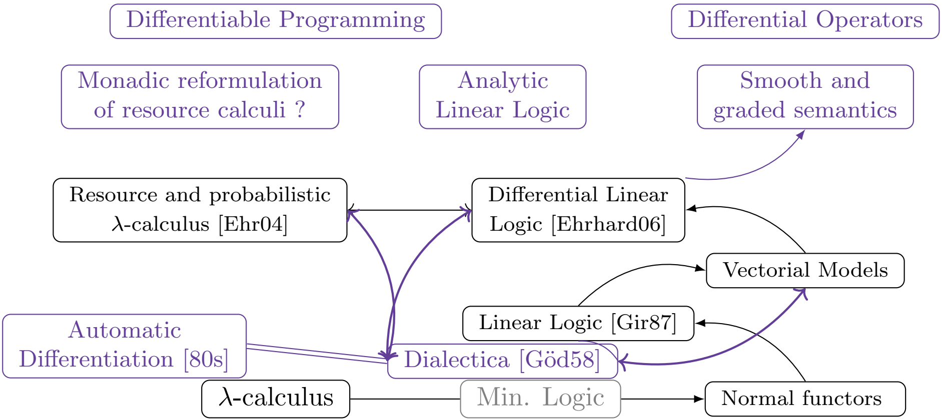

Perspectives

Quantitative Semantics: Approximating functions by Polynomials

- Taylor: Orthogonal Basis = $\{X^n\}$
  - recurrence : $T_{n+1} = XT_n$
  - composition $T_n \circ T_m = T_{nm}$

- Other Bases ? Chebychev ?
  - recurrence $T_{n+2} = 2T_{n+1} - T_n$
  - composition $T_n \circ T_m = T_{nm}$

- Characterization of approximation and orthogonality ?

Grading with partial differential operators

Grading by Linear Partial Differential Equations with constant coef.

$$[\![!_{\mathbf{D}} A]\!] := D((\mathcal{C}^\infty([\![A]\!], \mathbb{K})') \qquad [\![?_{\mathbf{D}} A]\!] := D^{-1}(\mathcal{C}^\infty([\![A']\!], \mathbb{K}))$$

parameters of the equations — solutions of the equations

$$D(\_) = f \qquad \phi \circ D = \_$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?_{Id} A}\, w \qquad \frac{\vdash \Gamma, f : ?_{D_1} A, g : ?_{D_2} A}{\vdash \Gamma, f.g : ?_{D_1 \circ D_2} A}\, c \qquad \frac{\vdash \Gamma, f : ?_{D_1} A}{\vdash \Gamma, f * E_{D_2} : ?_{D_1 \circ D_2} A}\, d$$

$$\frac{}{\vdash \delta_0 : !_{Id} A}\, \bar{w} \qquad \frac{\vdash \Gamma, \phi : !_{D_1} A \qquad \vdash \Delta, \psi : !_{D_2} A}{\vdash \Gamma, \Delta, \psi * \phi : !_{D_1 \circ D_2} A}\, \bar{c} \qquad \frac{\vdash \Gamma, \phi : !_{D_1} A}{\vdash \Gamma, \phi \circ D_2 : !_{D_1 \circ D_2} A}\, \bar{d}$$

Monoid: $(\mathrm{D}, \circ, Id)$

Breuvart, K. Mirwasser, 2023

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For D an LPDOcc: $(\phi \circ D) * \psi = (\phi * \psi) \circ D \qquad D(E_D * f) = f$

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?_{Id} A}\, w \qquad \frac{\vdash \Gamma, f : ?_{D_1} A, g : ?_{D_2} A}{\vdash \Gamma, f.g : ?_{D_1 \circ D_2} A}\, c \qquad \frac{\vdash \Gamma, f : ?_{D_1} A}{\vdash \Gamma, f * E_{D_2} : ?_{D_1 \circ D_2} A}\, d$$

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The computational content of differentiation.

The codereliction of differential proof nets: In terms of polarity in linear logic [23], the $\forall$-$\to$-free constraint characterizes the formulas of intuitionistic logic that can be built only from positive connectives ($\oplus$, $\otimes$, 0, 1, !) and the why-not connective ("?"). In this framework, Markov's principle expresses that from such a $\forall$-$\to$-free formula A (e.g. $? \oplus_x (?A(x) \otimes ?B(x))$) where the presence of "?" indicates that the proof possibly used weakening (`efq` or `throw`) or contraction (`catch`), a linear proof of A purged from the occurrences of its "?" connective can be extracted (meaning for the example above a proof of $\oplus_x(A(x) \otimes B(x))$). Interestingly, the removal of the "?", i.e. the steps from $?P$ to P, correspond to applying the codereliction rule of differential proof nets [24].

Differentiation : $(?P = (P \multimap \bot) \Rightarrow \bot) \to ((P \multimap \bot) \multimap \bot) \equiv P)$

Hugo Herbelin, "An intuitionistic logic that proves Markov's principle", LICS '10 .

This can also be witnessed by identifying the computational content of Dialectica as a CPS style differential λ-calculus.[PMP,K 22]

Open questions

- $\bar{\mathsf{p}}$ and p do not interact well: cut-elimination ?
- More intricate differential operators semi-rings? Higher-order methods ? Can we embed approximate resolution methods in the sequent calculus ?
- Can we express resolution methods in differential λ-calculi ?
- Can we make the categorical semantics of differentiation closer to the one of type theory ?

Conclusion

Take away

- The semantics of λ-calculus is not as much about discrete structures than about approximating continuous ones.
- The notion of linear type $\_ \multimap \_$ has been influential in functional programming. Let's now make use of the distribution type $!$, which internalizes external transformations on programs.
- Functional analysis and functional programming might enrich each other: the former gives the latter **new concepts**, the latter gives the former **new structures**.

Thank you for listening!