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TILINGS, SUBSTITUTION SYSTEMS AND DYNAMICAL SYSTEMS GENERATED BY THEM

By

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Abstract. The object of this work is to study the properties of dynamical systems defined by tilings. A connection to symbolic dynamical systems defined by one- and two-dimensional substitution systems is shown. This is used in particular to show the existence of a tiling system such that its corresponding dynamical system is minimal and topological weakly mixing. We remark that for one-dimensional tilings the dynamical system always contains periodic points.

§0. Introduction

The interest in questions concerning non-periodic tilings of the plane stems from the work of Hao Wang [7]. Wang, in connection with some decision problems of the predicate calculus, was interested in the decidability of the following problem:

The Tiling Problem. Given a finite set of unit square domino tiles with certain adjacency rules (called a *tiling system*), is there a tiling of the entire plane using copies of these tiles?

Wang has shown that this problem is related to the following question: Given a tiling system, such that it is possible to tile the entire plane with its tiles, is there a periodic tiling of the plane using these tiles?

Wang proved that an affirmative answer to the latter question would imply the decidability of the Tiling Problem. Wang conjectured that the answer is indeed affirmative.

This conjecture was disproved by Berger [1], who both proved the undecidability of the Tiling Problem and gave a complicated construction of a system of tiles that admits only non-periodic tilings of the plane.

Later Robinson [6] gave a construction of a smaller set of tiles with only non-periodic tilings of the plane and also a simpler proof of the undecidability of the Tiling Problem.

Notice that, given a tiling system, we may consider the set of all tilings of the entire plane as a topological space and there exists a natural action of $\mathbf{Z}^2$ on it by translations. Thus we have a dynamical system. This dynamical system is a two-dimensional symbolic dynamical system of finite type. The existence of a periodic tiling corresponds to the existence of a finite orbit in this system. We remark that a one-dimensional symbolic dynamical system of finite type always contains a periodic point. Thus the constructions of Berger [1] and Robinson [6] show a difference between the two- and one-dimensional cases. However, when one examines the dynamical system corresponding to Robinson's tiling system it is seen that it is almost periodic.

The object of this work is to study the properties of dynamical systems defined by tiling systems; mainly to check what classes of dynamical systems may be realized by tiling systems. In particular, we check whether there exist tiling systems such that their dynamical systems (and their minimal subsystems) are not almost periodic.

We define two-dimensional substitution systems, which are a generalization of the known one-dimensional substitution systems, and their dynamical systems. For a certain class of two-dimensional substitution systems we show how to construct a tiling system such that the dynamical system corresponding to the tiling system is isomorphic to the one defined by the substitution system. Then using results about one-dimensional substition systems we are able to show the existence of tiling systems with different dynamical properties. In particular, a tiling system with a dynamical system that is minimal and topological weakly mixing (hence not almost periodic) is constructed.

We remark that the construction of the tiling system is based on the ideas of the tiling system constructed by Robinson [6].

Section 1 contains background and definitions concerning symbolic dynamical systems, tiling systems and substitution systems. Section 2 describes the construction of a tiling system for a given two-dimensional substitution system. Section 3 examines the structure of the possible tilings of the plane by the tiles constructed in Section 2. In Section 4, the relations between the dynamical systems of the tiling system and the substitution system are proved. Section 5 deals with a more restricted class of substitution systems. Stronger relations between the dynamical systems are proved for this class. Section 6 describes modifications needed to deal with products of one-dimensional substitution systems. These enable us to represent many one-dimensional dynamical systems defined by one-dimensional substitution systems in dynamical systems defined by two-dimensional tiling systems. Section 7 contains several examples. Section 8 contains a brief description of a new proof of the undecidability of the Tiling Problem based on our construction of tiling systems.

§1. Symbolic dynamical systems

In this work we are interested in one- and two-dimensional symbolic dynamical systems. Let S be a finite set of symbols endowed with the discrete topology. A one-dimensional symbolic dynamical system is a pair $(\Omega, \mathbf{Z})$ where Ω is a closed subset of the compact space $S^{\mathbf{Z}}$ (with the product topology) invariant under the action of $\mathbf{Z}$ by translations. A two-dimensional symbolic dynamical system is a pair $(\Lambda, \mathbf{Z}^2)$ where $\Lambda \subset S^{\mathbf{Z}^2}$ is a closed set invariant under the action of $\mathbf{Z}^2$ by translations.

A symbolic dynamical system is determined by the closed set $\Omega \subset S^{\mathbf{Z}}$ $(\Lambda \subset S^{\mathbf{Z}^2})$. This set Ω can be characterized by the set U of all the finite words in symbols of S appearing as subwords of elements of Ω (in the case of two-dimensional systems, U is the collection of all rectangular blocks appearing in it). Given U we can reconstruct Ω (Λ). The elements of U are called *legal words* (or *blocks*). Since we will deal only with symbolic dynamical systems we will omit the word "symbolic".

Dynamical systems of finite type and tilings. Let C be a collection of words of length k over S. Define U to be the set of all finite words over S such that any subword of them of length k belongs to C. The symbolic dynamical system defined by U is called a one-dimensional symbolic dynamical system *of finite type*. Similarly, a two-dimensional dynamical system of finite type is determined by a collection C of $m \times n$ rectangular blocks where U is the collection of all finite blocks such that every subblock of size $m \times n$ of them belongs to C. We will use the term "*legal word* (*block*)" to refer to the elements of C and U.

Tiling systems. A (geometrical) *tiling system* is a finite collection of tiles in $\mathbf{R}^2$ such that every tile is a homeomorphic image of the unit disk. A tiling of the plane is a cover of the plane by sets, each of which is a translation of some tile, such that the interiors of the sets are disjoint.

A two-dimensional symbolic dynamical system of finite type determined by a set C of 2×2 blocks over a finite set of symbols S is said to satisfy the tiling condition if:

The tiling condition

1. Whenever $a\,b$, $a\,c$, $d\,c$ are legal adjacencies, then so is $d\,b$.
2. Whenever ${b \atop a}$, ${c \atop a}$, ${c \atop d}$ are legal adjacencies, then so is ${b \atop d}$.

A two-dimensional symbolic dynamical system satisfying the tiling condition may be "realized" geometrically by defining for each symbol $x \in S$ a "geometrical" tile which is a unit square with small perturbations allowed at the mid-third of every edge such that two symbols may be adjacent if and only if the corresponding tiles may be adjacent (Fig. 1).

Starting from a general two-dimensional symbolic dynamical system of finite type, defined by a set C of $n \times m$ blocks over a finite set of symbols S, we can construct an isomorphic tiling system. This is done as follows:

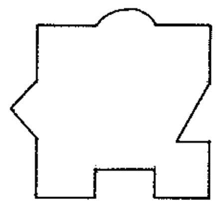

Fig. 1.

Define a new set of symbols consisting of all the $m \times n$ blocks over S appearing in C. Define the legal adjacencies as follows: two such new symbols may be horizontally or vertically adjacent if the corresponding $m \times n$ blocks are such that the left $m-1$ columns of the one on the right are the right $m-1$ columns of the other, or the lower $n-1$ rows of the upper one are the upper $n-1$ rows of the other.

It is easily checked that this system satisfies the tiling condition and may be realized as a geometrical tiling system, and that the two dynamical systems are isomorphic. Based on this observation, we will use the notion "tiling" to denote any symbolic dynamical system of finite type.

Substitution Systems

One-dimensional substitution systems

Definitions. (1) A one-dimensional substitution system is a pair $(\mathscr{A}, \mathscr{P})$ where:

$\mathscr{A}$ — The alphabet, a finite set of symbols or letters.

$\mathscr{P}$ — The derivation rules, a finite set of rules of the form: $a \to x_1 \cdots x_k$, $a, x_i \in \mathscr{A}$. Such a rule will be said to belong to a. k is the length of this rule.

(2) A substitution system will be called *deterministic* if for every letter there is at most one rule belonging to it.

(3) A one-dimensional substitution system is called *of size r* if all its rules have length r.

One-dimensional substitution systems will serve as a means of defining one-dimensional symbolic dynamical systems $(\Omega, \mathbf{Z})$. We have to define the set of words derived by the system.

Definition. Let $\alpha \in \mathscr{A}^*$ be a word $\alpha = a_1 \cdots a_n$, $a_i \in \mathscr{A}$. Define the collection of words derived from α:

$$\mathscr{D}(\alpha) = \left\{ w \;\middle|\; \begin{array}{l} w \text{ may be obtained from } \alpha \text{ by replacing each letter } a_i \\ \text{of } \alpha \text{ by the right side of some rule belonging to it.} \end{array} \right\}$$

Define

$$V_0 = \mathscr{A}, \quad V_{n+1} = \bigcup_{\alpha \in V_n} \mathscr{D}(\alpha), \quad V = \bigcup_{n \geq 0} V_n.$$

The set U of all the subwords of words in V is used to define the closed invariant set $\Omega \subset \mathscr{A}^{\mathbf{Z}}$. $(\Omega, \mathbf{Z})$ is the dynamical system defined by $(\mathscr{A}, \mathscr{P})$. The derivation process of a word $\alpha \in V$ may be described by a derivation tree of the form shown in Fig. 2. In this tree we find that a underwent the substitution $a \to x_1 x_2 \cdots x_k$ and x_1 underwent the substitution $x_1 \to y_1 y_2 \cdots y_l$, etc.

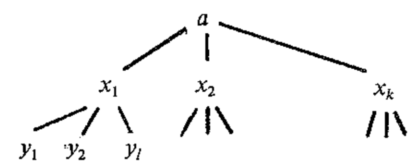

Fig. 2.

Two-dimensional substitution systems

Definitions. (1) A two-dimensional substitution system is a pair $(\mathscr{A}, \mathscr{P})$ where:

$\mathscr{A}$ — the alphabet, a finite set of symbols called letters.

$\mathscr{P}$ — a finite set of derivation rules of the form

$$a \to \begin{array}{ccc} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{array} \qquad a, x_{ij} \in \mathscr{A}$$

k is called the *height* of the rule and l its *width*. Such a rule is said to *belong to a*.

(2) A substitution system is called *deterministic* if for any $a \in \mathscr{A}$ there is at most one corresponding rule.

(3) A substitution system is called *of size* $k \times l$ if all its rules are of the form

$$a \to \begin{array}{ccc} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{array}$$

with k, l fixed.

In order to define the two-dimensional symbolic dynamical system generated by a two-dimensional substitution system, we have to define the set of all blocks (two-dimensional arrays) derived by the substitution system.

Definition. Let

$$\alpha = \begin{matrix} x_{n1} & \cdots & x_{nm} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1m} \end{matrix} \qquad \text{be a block, } x_{ij} = \mathscr{A}.$$

A *legal derivation* from α is choosing for every x_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ a derivation rule belonging to it and replacing every x_{ij} by the right-hand side of the chosen derivation rule. This process of choosing must be done so that for all the letters x_{ij}, $1 \leqq j \leqq m$ in one row, the derivation rules chosen all have the same height. For all the letters x_{ij}, $1 \leqq i \leqq n$ in one column, the derivation rules chosen all have the same width. These requirements ensure that for a block α, if it is legally derived, we get a rectangular block β. This process of replacing each letter according to some derivation rule is called *blowing up*. As in the one-dimensional case, we define the collection of blocks generated by the system to be the collection of all blocks for which there is a finite legal derivation sequence from a single letter. As in the one-dimensional case, there is a derivation tree for any block generated by the substitution system. Define the set of all blocks appearing as subblocks of a block generated by $(\mathscr{A}, \mathscr{P})$ to be the set of *legal* blocks. The dynamical system $(\Omega, \mathbf{Z}^2)$ defined by the substitution system $(\mathscr{A}, \mathscr{P})$ is obtained by setting $\Omega = \Omega(\mathscr{A}, \mathscr{P})$ to be the set of all the elements of $\mathscr{A}^{\mathbf{Z}^2}$ such that any finite subblock of them is a legal block.

Definition. A two-dimensional substitution system $(\mathscr{A}, \mathscr{P})$ will be called *of type A or having property A* if for any block U generated by it

$$U = (u_{ij})_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq m}}$$

and for any quadruple

$$L = \begin{pmatrix} u_{i+1,j} & u_{i+1,j+1} \\ u_{i,j} & u_{i,j+1} \end{pmatrix}$$

appearing in U, the following holds:

If there is a sequence of legal derivations $L^0 = L, L^1, \ldots, L^r$ where L^{i+1} is legally derived from L^i, then there is a sequence of legal derivations $U^0 = U, U^1, U^2, \ldots, U^r$ such that U^{i+1} is legally derived from U^i and, in this derivation sequence, looking at the blocks derived from L, we see the derivation sequence $L^0, L^1, \ldots, L^r$.

Definition. For a two-dimensional substitution system $(\mathscr{A}, \mathscr{P})$, define the collection of *legal quadruples* $Q = Q(\mathscr{A}, \mathscr{P})$ by

$$Q = \left\{ \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \;\middle|\; a, b, c, d \in \mathscr{A} \text{ there is a block generated by } (\mathscr{A}, \mathscr{P}) \text{ containing } \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \right\}.$$

Clearly Q is finite.

§2. Construction of a tiling system for a two-dimensional substitution system

We will describe in this section a procedure for constructing for a given two-dimensional substitution system $(\mathscr{A}, \mathscr{P})$ a tiling system $(\mathscr{T}, \mathscr{N})$, where $\mathscr{T}$ is the set of tiles and $\mathscr{N}$ the set of adjacency rules. The substitution system should satisfy the following condition:

(1) Every derivation rule

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

of the system $(\mathscr{A}, \mathscr{P})$ satisfies $k \geqq 2$, $l \geqq 2$.

Let $Q = Q(\mathscr{A}, \mathscr{P})$ denote the set of legal quadruples of the substitution system $(\mathscr{A}, \mathscr{P})$.

The Tiles $\mathscr{T}$

The tiles $\mathscr{T}$ of the tiling system will belong to several classes.

Class I: letter tiles. This class consists of tiles in which letters of $\mathscr{A}$ are written. The tiles of this class will be of two types: type 1 and type 2. A tile in this class (of both types) will hold, in addition to the letter, the following information:

(1) The derivation rule (from $\mathscr{P}$) in which the letter appears in the right side of the rule.

(2) The letter position in this rule.

The information in (1) and (2) must be consistent with the letter of the tile.

(3) A tile which, according to the information in (1) and (2), is positioned at one of the four corners of the derivation rule will designate also a legal quadruple belonging to Q such that:

Let "a" be the letter from which the rule is derived. If the tile is the upper left corner of the derivation rule, then "a" is the lower right letter of its quadruple. If the tile is the upper right corner of the derivation rule, then "a" is the lower left letter of the quadruple. If the tile is the lower left corner of the derivation rule, then "a" is the upper right letter of the quadruple. If the tile is the lower right corner of the derivation rule, then "a" is the upper left letter of the quadruple.

Examples. Are shown in Fig. 3.

0
position (2,3)
in the rule

$$1 \longrightarrow \begin{matrix} 1\,1\,1\,1 \\ 0\,1\,0\,1 \\ 0\,1\,1\,1 \end{matrix}$$

1
position (3,3)
in the rule

$$0 \longrightarrow \begin{matrix} 0\,0\,1 \\ 0\,0\,0 \\ 0\,0\,0 \end{matrix}$$

quadruple $\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}$

Fig. 3.

Class II: arrows, junctions and lines tiles. Each tile of this class contains several items of information. The items are of the following kinds:

(1) Line.
(2) Arrow.
(3) Junction.
(4) "A", "B", "R", "L".

A tile of this class contains some combination of items of these kinds. The allowed combinations will be explained after the description of each of the above kinds.

General remarks

(a) Items of the kind "Line", "Arrow" or "Junction" contain a line, arrow or junction and some information. The line, arrow or junction is used to give "orientation" to the information associated with it. We describe such situations by saying that the line, arrow or junction holds this information.

(b) We refer to items like the above as "tiles". Thus by "line tile" we mean a tile of this class containing a line.

Line. A line is either horizontal or vertical. The information of a line may be one of the following:

(1) For a horizontal line: "I link positions (i, j) and $(i, j+1)$ in the rule

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{"}$$

where $1 \leqq i \leqq k$, $1 \leqq j \leqq l-1$, except for the pairs (1, 1) and $(k, 1)$.

(2) For a horizontal line: "I link positions $(i, 1)$ in the rule

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

and position (i, m) in the rule

$$b \to \begin{matrix} y_{k1} & \cdots & x_{km} \\ \vdots & & \vdots \\ y_{11} & \cdots & x_{1m} \end{matrix} \text{"}$$

where $1 \leqq i \leqq k$.

(3) A line, as in (2), such that $i = 1$ or k holds also a quadruple such that for $i = 1$ the upper two letters of it are "ba" and, for $i = k$, the lower two letters of it are "ba".

(4) For a vertical line: "I link positions (i, j) and $(i+1, j)$ in the rule

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{"}$$

where $1 \leqq i \leqq k-1$, $1 \leqq j \leqq l$ except for the pairs (1, 1) and $(1, l)$.

(5) For a vertical line: "I link position $(1, j)$ in the rule

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

and position (r, j) in the rule

$$c \to \begin{matrix} z_{r1} & \cdots & z_{rl} \\ \vdots & & \vdots \\ z_{11} & \cdots & z_{1l} \end{matrix} \text{"}$$

where $1 \leqq j \leqq l$.

(6) A vertical line, as in (5) such that $j = 1$ or l holds also a quadruple such that, for $j = 1$, the right half of the quadruple is "$\begin{smallmatrix} a \\ c \end{smallmatrix}$" and for $j = l$ the left half of the quadruple is "$\begin{smallmatrix} a \\ c \end{smallmatrix}$".

The lines described in (1) and (4) are called interior lines. The lines described in (2), (3), (5) and (6) are called exterior lines.

Arrows. There are two kinds of arrows:

Simple arrow. An arrow of this kind holds the following information:

(1) A horizontal right directed arrow: "I originate at position $(i, 1)$ in the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{''}$$

where $i = 1$ or k.

(2) A horizontal left directed arrow. "I originate at position $(i, 2)$ in the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{''}$$

where $i = 1$ or k.

(3) A vertical upward directed arrow: "I originate at position $(1, j)$ in the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{''}$$

where $j = 1$ or l.

(4) A vertical downward directed arrow: "I originate at position $(2, j)$ in the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \text{''}$$

where $j = 1$ or l.

Combined arrows. This kind of arrow holds the following information:

(1) A vertical upward directed arrow: "I originate at the junction between positions (1, 1) and (1, 2) of the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l}. \end{matrix} \text{''}$$

(2) A vertical downward directed arrow: "I originate at the junction between positions $(k, 1)$ and $(k, 2)$ of the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l}. \end{matrix} \text{''}$$

(3) A horizontal left directed arrow: "I originate at the junction between positions (1, 1) and (2, 1) of the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l}. \end{matrix} \text{''}$$

(4) A horizontal right directed arrow: "I originate at the junction between positions $(1, l)$ and $(2, l)$ of the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l}. \end{matrix} \text{''}$$

(5) Any combined arrow holds also a counter counting interior lines crossed.

The Junction Tiles. These tiles contain three arrows; two meeting simple arrows ending in this tile and one combined arrow perpendicular to the other two starting in it. There are four types, shown in Figs. 4–7.

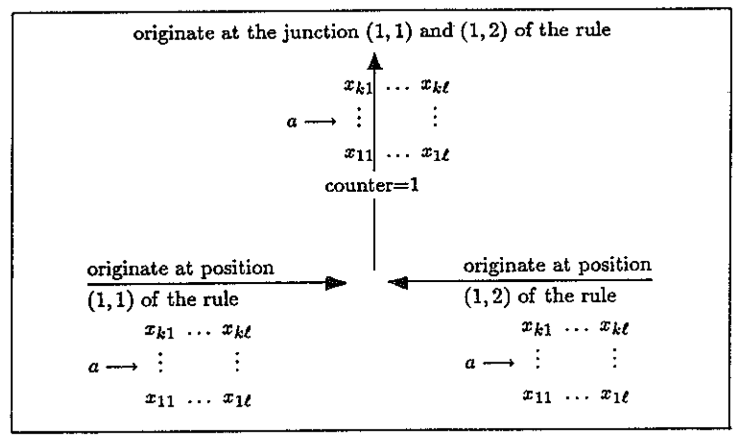

Fig. 4.

"A", "B", "R", "L" Tiles

A tile containing "A", "B", "R" or "L" always contains also an exterior line. There are four kinds of "A", "B", "R", "L" tiles.

(1) A — "Above". Here the exterior line is vertical.
"A" means that this tile is above the intersection of the line with a horizontal combined arrow.

(2) B — "Below". Here the exterior line is vertical.

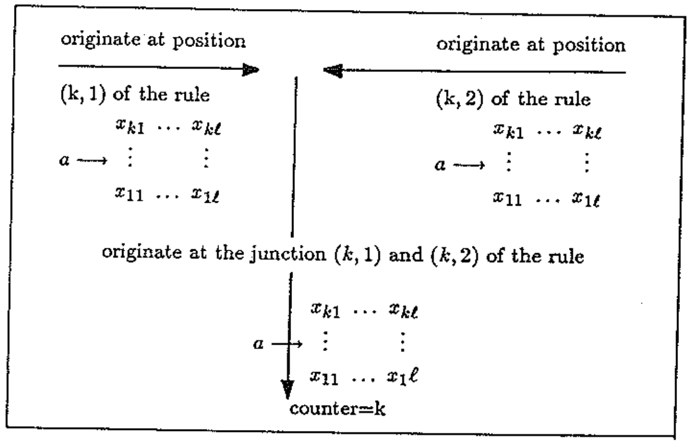

Fig. 5.

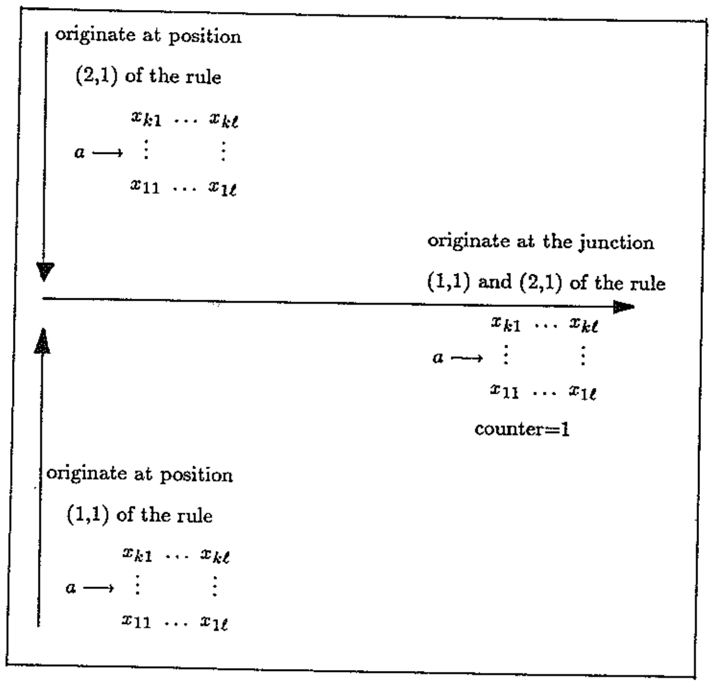

Fig. 6.

"B" means that this tile is below the intersection of the line with a horizontal combined arrow.

(3) R — "Right". Here the exterior line is horizontal.
"R" means that this tile is to the right of the intersection of the line with a vertical combined arrow.

(4) L — "Left". Here the exterior line is horizontal.

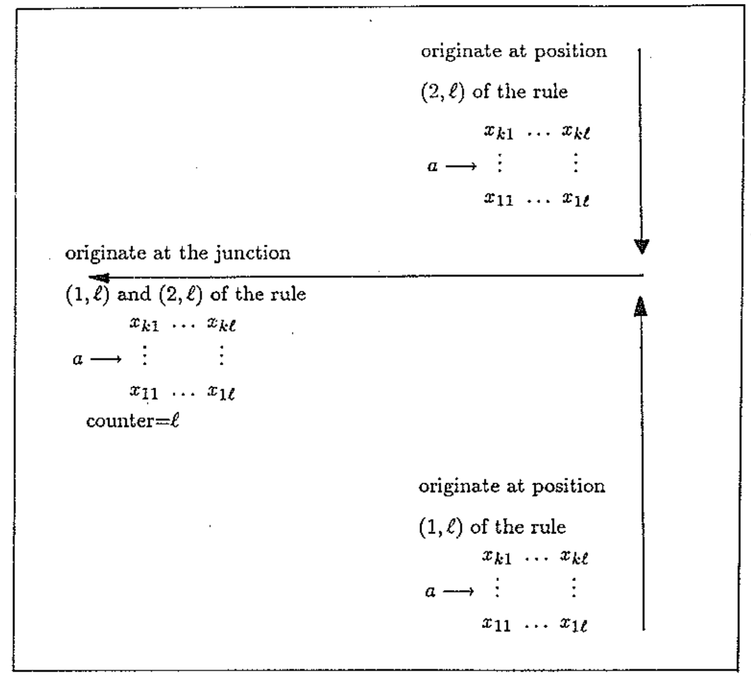

Fig. 7.

"L" means that this tile is to the left of the intersection of the line with a vertical combined arrow.

Combination of "A", "B", "R", "L" tiles

A tile may contain up to two of "A", "B", "R" or "L". The possible pairs are:

(1) "A" and "R",
(2) "A" and "L",
(3) "B" and "R",
(4) "B" and "L".

Combinations of lines, arrows and junctions in one tile

As mentioned earlier, a tile may contain more than one arrow or line, according to the following rules:

(1) No more than one arrow in each direction.
(2) No two parallel lines in a tile.
(3) Two perpendicular lines in a tile are possible only in one of the following cases:
 (a) One, at least, of the lines is an exterior line.
 (b) The horizontal line links positions in columns 1 and 2 of a rule. The vertical line links the letter from which the rule of the horizontal line is derived.

(c) The vertical line links positions in rows 1 and 2 of a rule. The horizontal line links the letter from which the rule of the vertical line is derived.

(4) A simple arrow and a line perpendicular to each other. This satisfies the same conditions as in (3) replacing one line by a simple arrow.

(5) A combined arrow and a line, perpendicular to one another, is possible only in one of the following cases:

(a) The line is an exterior line.

(b) A horizontal line linking positions in columns 1 and 2 at height $i \geqq 2$ in the rule. The combined arrow is directed downwards and its counter value is i.

(c) A vertical line linking positions in rows 1 and 2 at column $j \geqq 2$ of the rule. The combined arrow is directed to the left and its counter value is j.

(6) A junction tile and a line linking a letter (in the direction of the combined arrow) from which the rule of the junction is derived.

(7) A line and an arrow parallel to each other must be one of the following cases:

(a) The arrow is a combined arrow and the line links in the direction of the arrow a letter from which the rule of the arrow is derived.

(b) The tile is a junction tile like case (6).

(8) Two parallel arrows. One of them must be a simple arrow and then the situation is like (7) but with "simple arrow" replacing "line".

(9) A tile containing a combined arrow *must* contain also a line or a simple arrow in opposite directions, such that the line or the simple arrow contains information like in (7) or (8).

Class III: blank tile. This tile doesn't contain any information and is used to "fill holes".

The Adjacency Rules $\mathcal{N}$

(A) Let α be a letter tile (of either type) such that the letter written in it is y and it is positioned at (i, j) in the rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix} \qquad (x_{ij} = y)$$

If $(i, j) \notin \{(1, 1), (1, 2), (1, l), (2, l), (2, 1), (k, 1), (k, 2)\}$ then its four neighbours are line tiles such that the lines are directed from this tile. The information in the lines is consistent with that of the letter tile α; that is, each of them links position (i, j) of rule

$$a \rightarrow \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

to the position that should be in its direction in the derivation rule, or to a neighbouring rule in the case that (i, j) is at the border of the rule.

If $(i, j) \in \{(1, 1), (1, 2), (1, l), (2, l), (2, 1), (k, 1), (k, 2)\}$, the neighbours are the same, except that one or two of them is a simple arrow (or a junction) instead of a line.

If $(i, j) \in \{(1, 1), (1, l), (k, 1), (k, l)\}$, that is, it is a corner tile, then α contains also a quadruple and the exterior lines adjacent to it contain this quadruple too.

An **Example** is shown in Fig. 8.

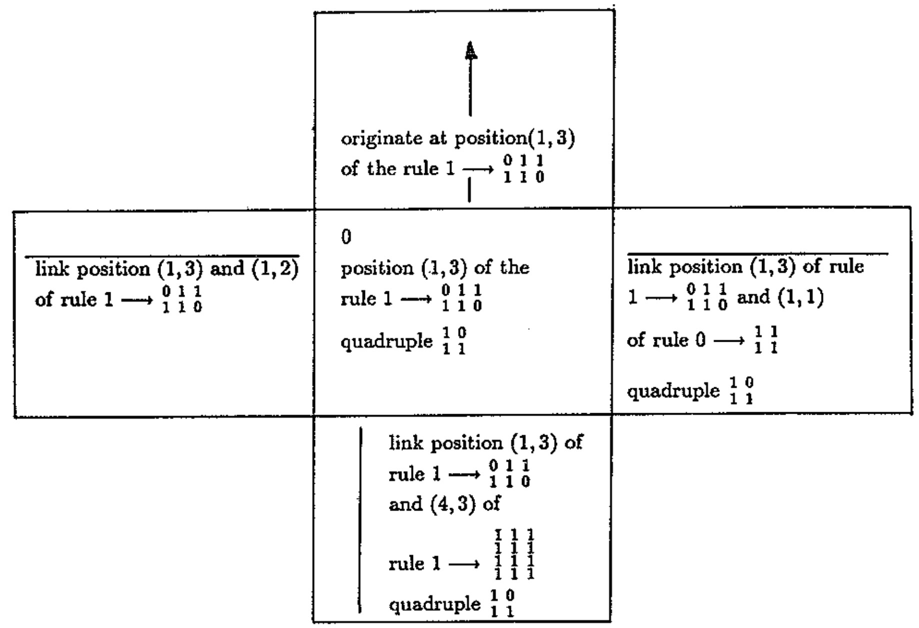

Fig. 8.

(B) A horizontal line tile must meet on its right and its left one of two possible tiles:

(1) A horizontal line tile containing the same information.

(2) A letter tile holding information consistent with that of the line.

The neighbours above and below a vertical line tile satisfy analogous requirements.

(C) A simple arrow tile must meet in the direction of the arrow a simple arrow, holding the same information, or a junction tile such that the arrow in its direction contains the same information.

In the opposite direction it meets a simple arrow with the same information or a letter tile holding information consistent with that of the simple arrow.

An **Example** is shown in Fig. 9.

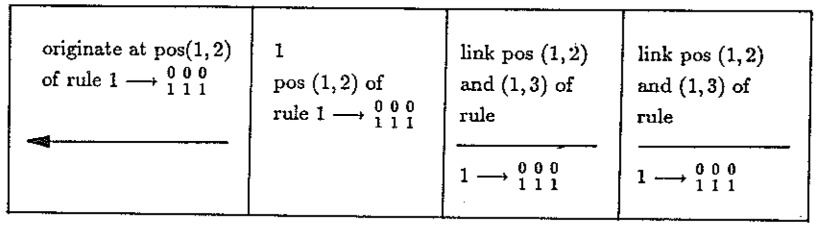

Fig. 9.

(D) A junction tile has the same adjacency rules on the side pointed by the combined arrow as a combined arrow tile. It has the same adjacency rules as a simple arrow on the two sides from which the simple arrows come.

(E.1) A combined arrow directed downwards or to the left with counter reading 2 meets in the direction of the arrow either a combined arrow containing the same information, or a letter tile of type 2 such that the letter written in it is the letter from which the derivation rule of the arrow is derived. If the counter is $\geqq 3$ then the tile meets in the direction of the arrow either a combined arrow with the same information or a combined arrow intersected by an interior line of that rule. In this case the combined arrow in the adjacent tile contains the same information except that the counter is decreased by 1. See also the paragraph describing the combination of arrows and lines in one tile.

If the rule of the combined arrow is of size $k \times l$, then for a horizontal combined arrow with counter $= l$ or a vertical combined arrow with counter $= k$, the tile adjacent to the tile in the direction opposite to the combined arrow may be either a combined arrow with the same information or a junction from which such a combined arrow tile starts. For a horizontal combined arrow with counter $j < l$ or a vertical combined arrow with counter $i < k$ the adjacent tile in the direction opposite to the direction of the combined arrow contains a combined arrow with the same information unless our tile has an interior line in it. In this case the adjacent tile contains a combined arrow with the same information except that the counter is increased by 1.

(E.2) A combined arrow directed upwards or to the right, meets in the direction of the arrow a combined arrow containing the same information, or a letter tile of type 2 such that the letter written in it is the letter from which the derivation rule of the arrow is derived. In the direction opposite to that of the arrow such a tile meets a combined arrow with the same information or a junction in which a combined arrow with the same information is generated.

(F) A letter tile of type 2 requires all its four neighbours to contain combined arrows (including junctions at which such combined arrows are generated) directed towards itself. And the letter written in the letter tile is the one from which the common derivation rule of the combined arrows is derived. The counters of the four arrows must be 1 for the arrows below and left of the tile and 2 for the arrows above and to the right of it.

An **Example** is given in Fig. 10.

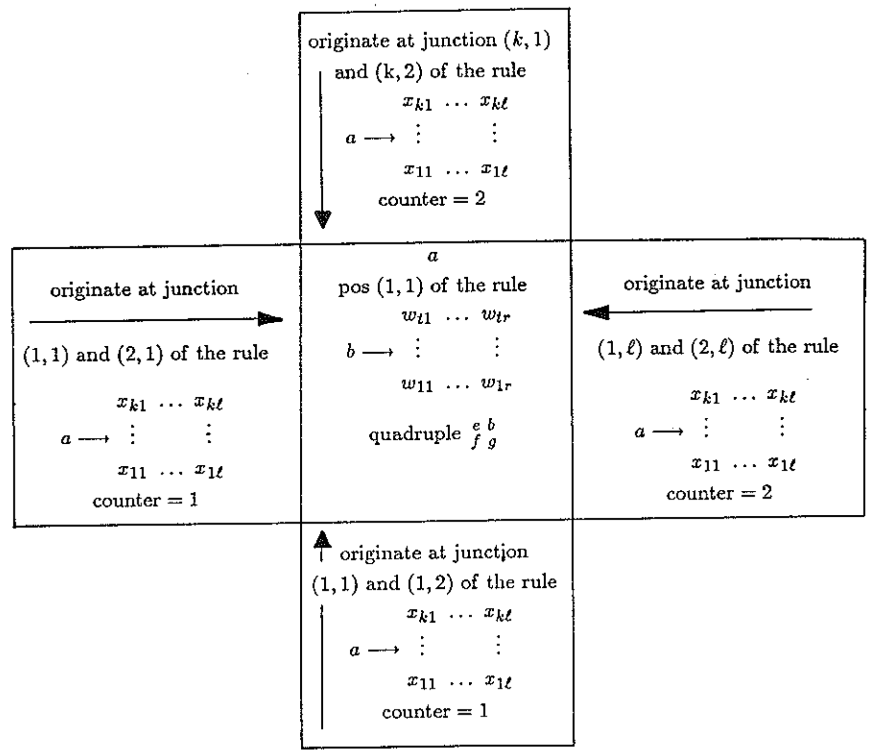

Fig. 10.

(G) A tile containing a horizontal combined arrow intersecting a vertical exterior line has above it either a letter tile of type 1 or a tile containing "A". Similarly it has below it either a letter tile of type 1 or a tile containing "B".

Analogously, a tile containing a vertical combined arrow intersecting a horizontal exterior line has on its right either a letter tile of type 1 or a tile containing "R". Also, it has on its left either a letter tile of type 1 or a tile containing "L".

(H) (1) A tile containing "A" has above it either a letter tile or a tile containing "A". It has below it either a tile containing the intersection of a horizontal combined arrow and a line, or a tile containing "A".

(2) A tile containing "B" has below it either a letter tile or a tile containing "B". It has above it either a tile containing the intersection of a horizontal combined arrow and a line, or a tile containing "B".

(3) A tile containing "L" has on its left either a letter tile or a tile containing "L". It has on its right either a tile containing the intersection of a vertical combined arrow and a line, or a tile containing "L".

(4) A tile containing "R" has on its right either a letter tile or a tile containing "R". It has on its left either a tile containing the intersection of a vertical combined arrow and a line, or a tile containing "R".

(I) Now we use the observation that every system of finite type is a tiling. Further, we require that in every square of 3×3 tiles, if we paint the letter tiles of type 1 black and all the other tiles white, we will see one of the patterns in Fig. 11.

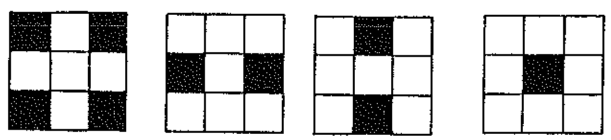

Fig. 11.

Notice that this rule ensures that the letter tiles of type 1 are arranged in a $2\mathbf{Z} \times 2\mathbf{Z}$ lattice.

(J) The blank tile may appear at any place where the above rules don't force the appearance of some other kind of tile.

This completes the definition of the tiling system built for a certain substitution system.

Terminology. We will use the term "a straight lattice" or simply "lattice" to denote a subset of $\mathbf{Z}^2$ of the form $U \times V$ where U, V are infinite subsets of $\mathbf{Z}$.

§3. $(\mathscr{T}, \mathscr{N})$-tilings

In order to determine the collection of all possible tilings by $(\mathscr{T}, \mathscr{N})$ we will look at the tiling as a process such that, at any stage of the process, the content of the tiles arranged in some straight lattice is revealed (determined). These tiles will all be letter tiles. We will see how their contents force a certain structure of a system of lines, arrows and junctions which induce restrictions on the contents of the tiles of the current lattice and determine the lattice of the next stage of the process. We will call the lattice determined at stage k "a k-level lattice".

As mentioned earlier, the letter tiles of type 1 are arranged in a straight lattice $2\mathbf{Z} \times 2\mathbf{Z}$. This is the 1-level lattice shown in Fig. 12.

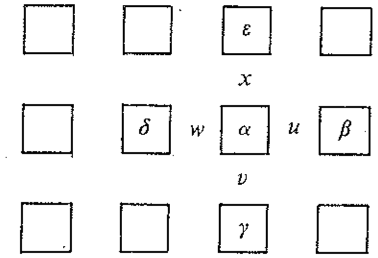

Fig. 12.

Let each of these letter tiles choose the information written in it. Look at any one of them, say α. By the adjacency rules its four neighbours u, v, w, x must be lines, arrows or junctions holding information about α. Notice that since the tiles of this lattice are arranged on $2\mathbf{Z} \times 2\mathbf{Z}$, the four neighbours of α meet β, γ, δ and ε respectively and these are letter tiles. It is easily seen from the adjacency rules of the tiling system that this forces a "synchronization" between α and its neighbours in the lattice. Suppose, for example, α contains the information "0, position (1, 3) of the rule $1 \to \begin{smallmatrix}0&1&1\\1&1&0\end{smallmatrix}$, quadruple $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$". This will force

(i) u to be an exterior line and β to be the lower left corner of some rule of height 2 with the same quadruple as α,

(ii) v to be an exterior line and γ the upper right corner of some derivation rule of width 3 with the same quadruple as α,

(iii) w to be an interior line and δ to be: "1, pos (1, 2) of the rule $1 \to \begin{smallmatrix}0&1&1\\1&1&0\end{smallmatrix}$",

(iv) x to be a junction between positions (1, 3) and (2, 3) of the rule $1 \to \begin{smallmatrix}0&1&1\\1&1&0\end{smallmatrix}$ and ε is "1, position (2, 3) of the rule $1 \to \begin{smallmatrix}0&1&1\\1&1&0\end{smallmatrix}$, quadruple $\begin{smallmatrix}*&*\\1&0\end{smallmatrix}$".

We see that all the tiles in the 1-level lattice are grouped in this way into rectangular blocks, each belonging to some derivation rule, and that these blocks are arranged in a lattice. Furthermore, looking in a row, all the tiles are at the same height in their rules, and looking at a column we see tiles in the same horizontal position in their rules. This is illustrated in Fig. 13 (dotted lines denote exterior lines).

Consider such a rectangular block belonging to some derivation rule. We see that there are four junctions such that the combined arrows in them point to one particular tile. Look, for example, at the junction at α in Fig. 13. Moving in the direction of the arrow we must see by the adjacency rules a combined arrow with the same information or a letter tile of type 2. Since if we continue in this direction we meet the opposite junction of the same rule, the sequence of arrows must terminate and meet a letter tile. We claim that this tile must be the tile at the intersection of the four arrows from the four junctions, in our example at δ. The reason is that the other tiles between the opposite junctions are either line tile, like γ, hence not a letter tile, or have, like β, a neighbour above it which is a line

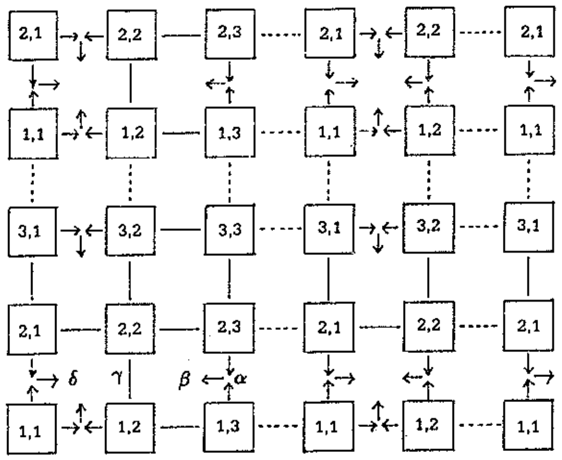

Fig. 13.

tile linking columns other than 1 and 2. By the adjacency rules, above a letter tile of type 2 there is a combined arrow directed towards it and no line tile as that above β may contain such an arrow.

We see that in any block belonging to some derivation rule there is one tile forced to be a letter tile of type 2 and the letter written in it is the letter from which that derivation rule is derived. From the "synchronization" of the tiles of the 1-level and of their blocks, it follows that the letter tiles of type 2 described above are arranged in a straight lattice. These tiles are the 2-level lattice. Notice also that the counter information along a combined arrow is determined by this process. The situation at this stage is as in Fig. 14.

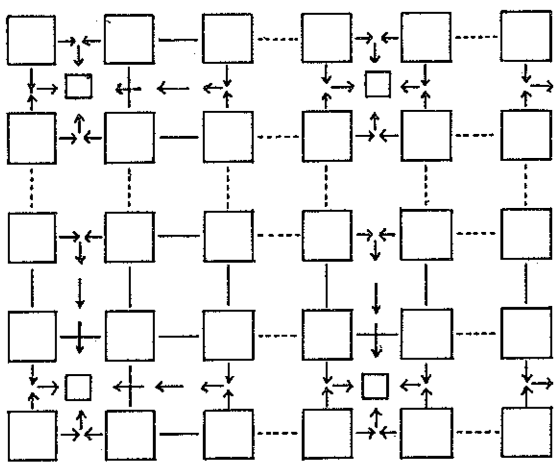

Fig. 14.

Definition. A rectangular block of tiles is called *saturated* if in no position in it can one add a letter tile, an arrow, a junction, a line or " "A", "B", "L", "R" " tile not already there.

Obviously, a single letter tile is saturated. We now wish to show how to continue the process of unravelling the tiles by induction. Our induction hypothesis is as follows:

Induction hypothesis. Assume we have already formed m lattices of letter tiles: level 1, level 2, ... till level m. The letter tiles of each level k, $1 \leqq k \leqq m-1$ are grouped into rectangular blocks each corresponding to some derivation rule. These blocks are arranged in a straight lattice. In the "convex hull" of each such block in a certain position there is a letter tile of the $(k+1)$-level lattice such that the letter written in it is the one from which the block of the k-level letter tiles containing it is derived. Moreover, for each letter tile of the k-level lattice $2 \leqq k \leqq m$ there corresponds a unique block of letter tiles of the $(k-1)$-level lattice derived from it by some derivation rule. Starting from a letter tile in the k-level lattice we can follow the sequence of derivations from it until we reach the 1-level lattice; we get a legal derivation tree of the substitution system. The letter tiles in the 1-level lattice we reached in such a way starting from some letter tile in the k-level lattice form a rectangular array. The convex hull of this rectangular array is a rectangular block caled "the block belonging (or corresponding) to the letter tile we started with".

We assume also that each block corresponding to a letter tile of the $(m-1)$-level lattice is saturated. (In case $m = 2$ this block is just the letter tile of the 1-level lattice and is clearly saturated.)

Further, we assume that looking at the rectangular arrays of letter tiles in the $(m-1)$-level lattice grouped according to the derivation rules, then between any two adjacent such arrays there is exactly one "free" row or column, where by "free" we mean that at this stage of the process only exterior lines appear in it, and these exterior lines have not intersected any combined arrow yet.

Furthermore, if we draw a picture of the situation where the blocks belonging to the tiles of the $(m-1)$-level are represented by rectangles, we see a picture like in Fig. 15.

The squares α, β are letter tiles of level m. The counter information of the combined arrows in the picture is fixed and each of them intersects the required number of interior lines. It is easy to check that the induction hypothesis holds for $m = 2$.

In the proof of the induction step we show how the tiles which are the letter tiles of the $(m+1)$-level lattice are determined, and how the information in them and other tiles is completed so that the induction hypothesis holds for $m+1$.

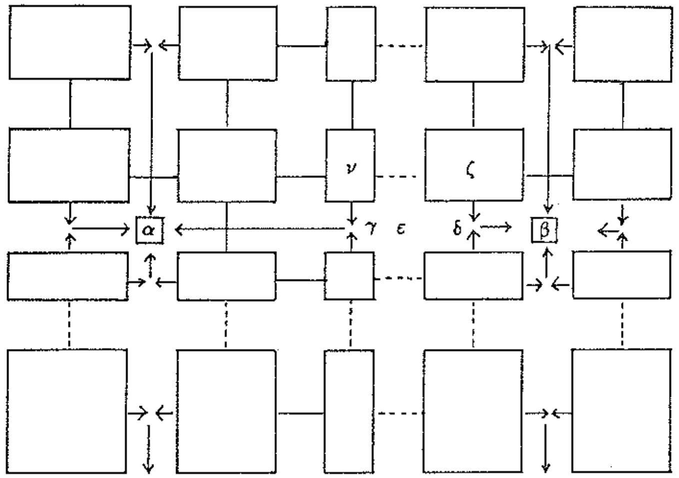

Fig. 15.

Suppose it holds for m. Each of the letter tiles of the m-level lattice has to complete the information in it consistently with the letter already in it. After this completion, four lines or arrows according to the letter's position in the rule it chose, start from the tile in the four directions. We want to show how the information they carry reaches the neighbouring letter tiles in the m level lattice. For simplicity of explanation, we show it for horizontal neighbours like α and β in the above picture (Fig. 15). Suppose first that α is position (i, j) in a rule such that to the right of α starts an interior line. Continuing to the right we must see a line of the same information or a letter tile corresponding to position $(i, j+1)$ in the same rule. Continuing to the right, starting at α, we eventually reach β. In β there is a letter tile, hence the sequence of line tiles starting to the right of α cannot continue indefinitely and must meet a letter tile corresponding to position $(i, j+1)$ at a position not beyond β. Let η be this tile. Since all the tiles between α and γ contain a combined arrow directed towards α, they cannot be letter tiles. Suppose that η is between γ and ε including ε. The letter tile must be of type 2 since all the letter tiles of type 1 are in the $2\mathbf{Z} \times 2\mathbf{Z}$ lattice of the first level and this row does not contain any of them. A letter tile of type 2 must have a combined arrow leading to it from every direction. Follow the one on the left of η backwards. We see a sequence of such rightward directed combined arrows. This sequence must end before reaching γ and it ends (actually starts) in a junction. This junction forces an arrow or a letter tile above it, but that tile is in a saturated block ν belonging to some $(m-1)$-level letter.

A similar argument following the sequence of combined arrows pointing to η from the right shows that η cannot be between ε and δ. η cannot be between δ and β since all the tiles there contain a combined arrow. Hence we see that the sequence of line tiles starting at α to the right must lead to β and is a line linking α

and β. This forces the synchronization of α and β. Notice that the case of an exterior line is completely analogous.

We analyze now the case in which α is position, say, (1,1) in a rule and then to the right of it starts a sequence of simple arrows. As before, because in β there is a letter tile, the sequence of arrows from α cannot continue indefinitely. Such a sequence must end in a horizontal junction tile. Let η be the position of this junction. Since the tiles between α and γ contain combined arrows, η cannot be in this range. η cannot be between γ and ε (excluding ε) since such a junction tile forces an arrow or a letter above it in ν and ν is saturated. (Notice that the tile it forces is not already in ν. This is seen by looking at the induction process.) For similar reasons η is not between ε and δ (excluding ε), and since the tiles between δ and β contain a combined arrow directed toward β it is not in that range too. Hence the junction must be in ε. Now to the right of such a junction we see a sequence of left directed simple arrows containing the information that they originate at position (1,2) of the same rule. Following this sequence of arrows we must reach a letter tile with consistent information (again the letter tile at β prevents the possibility of an infinite sequence of arrows). This letter tile cannot be positioned between ε and δ since it must be a letter tile of type 2 and hence will force a combined arrow above it in ζ which is a saturated block. This letter tile cannot be positioned between δ and β since the tiles there contain combined arrows. Hence this letter tile must be the one in β and this forces the synchronization of the tiles in α and β.

The foregoing discussion, with obvious changes, applies also to arrows and lines starting from α in any other direction. This shows that the letter tiles of the m-level lattice are grouped into rectangular arrays each corresponding to some derivation rule. These arrays are arranged in a straight lattice and between any two adjacent such arrays there is exactly one free column or row.

Take now a letter tile α of the m-level lattice, and look at the sequence of derivations from this letter until it reaches the 1-level lattice. Look at the "convex-hull" of these tiles. It forms a rectangular block of the form (Fig. 16).

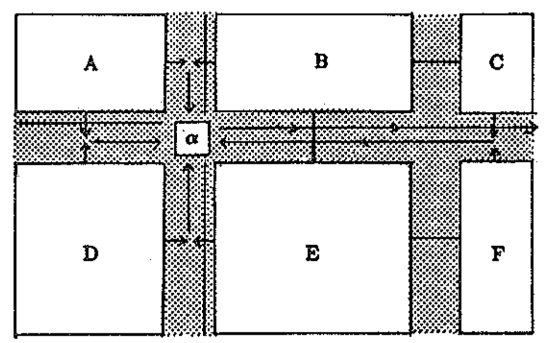

Fig. 16.

Suppose for simplicity that the rule derived from the letter in α is of size 2×3, where A, B, C, D, E, F are saturated blocks belonging to the letters in the $(m-1)$-level lattice derived from the letter in α.

It is easily seen by the induction hypothesis and by the rules of the tiling that the only seemingly possible places to add some arrow, line or letter are in the shaded areas. To see that the rectangle is saturated, notice first that no letter tile of type 1 may be added to it since these tiles form a $2\mathbf{Z} \times 2\mathbf{Z}$ lattice. Next, notice that adding a letter tile of type 2 or a combined arrow tile or a junction tile to the shaded area forces a sequence of such tiles that eventually try to penetrate one of the saturated blocks A, B, C, D, E or F or to place such a tile on some existing combined arrow tile or an interior line or simple arrow which is impossible. The case of an interior line tile or simple arrow is handled by observing that it forces a sequence of interior line tiles which may end in a letter tile. By the previous discussion it cannot reach a letter tile in the shaded area. Hence the sequence of line tiles forced by it either tries to penetrate A, B, C, D, E or F which is impossible, or it crosses one of the four combined arrows which is also impossible, either because the specific interior line cannot cross a combined arrow like the one in the rule or because the counter information along the combined arrow is already fixed and a new intersection will lead to inconsistency. We have to show the impossibility of the addition of some exterior line. To prove the impossibility of such addition we have to consider two neighbouring blocks belonging to adjacent m-level lattice tiles. We illustrate the proof for a vertical exterior line. Since no letter tile may be added to the block and since A, B, C, D, E, F are saturated, the vertical line tile must force a sequence forming a vertical line crossing the whole block and intersecting the combined arrow as in the diagram in Fig. 17.

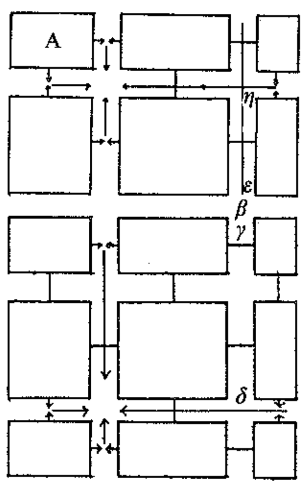

Fig. 17.

Consider the tile at β. It may be either a letter tile of type 2, in which case it forces a combined arrow in ε, which is impossible, or an exterior vertical line. In the latter case the tile in γ is either a letter tile or an exterior vertical line. The former is impossible by the same reasoning as above, applied to the adjacent block. In the latter case the same line of reasoning shows that it forces a sequence of vertical exterior line tiles crossing the horizontal combined arrow in δ. Now notice that we have a vertical exterior line crossing two combined arrows. But the crossing forces along the line segment between η and δ the occurrence of the symbols "A" and "B" and no tile can contain both of them; hence a contradiction. The last case to check is the addition of some "A", "B", "L", "R" tile. Such an addition is impossible by the fact that such a tile necessarily contains some exterior line too. If this exterior line already exists, it either already contains an "A", "B", "L", "R" tile or the exterior line links two letter tiles without intersecting a combined arrow. Such a situation is impossible, since it forces the same symbol "A", "B", "L" or "R" along it and we get an illegal adjacency of an "A", "B", "L", "R" tile and a letter tile.

This completes the proof of the saturation of the block belonging to a tile of the m-level lattice. It should be noticed that we have tacitly used in the above discussion the fact that if we have two adjacent tiles α and β and some content x in α forces some content y in β, then the content y in β forces either content x in α or some content z in α that cannot appear together with x in one tile.

We now concentrate on some rectangular group of m-level lattice tiles belonging to some derivation rule. We have the following picture where the rectangles denote the saturated block belonging to the m-level lattice tiles belonging to this derivation rule (Fig. 18). (Suppose for example the rule's size is 3×3.)

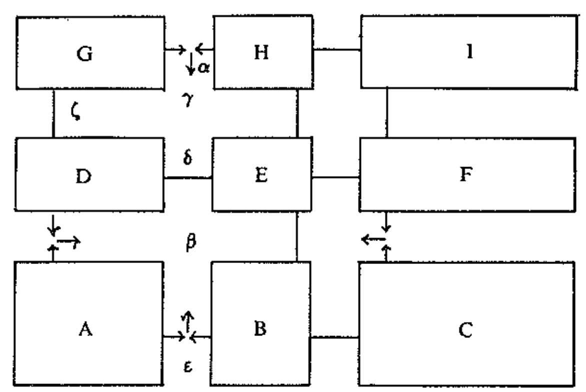

Fig. 18.

The four junction tiles are placed in a row and a column and the combined arrows are directed towards the tile β common to the row and column. In order to complete the induction step we have to show that there are sequences of combined arrows starting from each of the four junctions leading to a letter tile of type 2 positioned at β containing the letter from which the rule is derived. We show this, for example, for the junction at α. The reasoning for the other three junctions is completely analogous. Starting from α going downwards we see a sequence of combined arrows such that each of them may be followed by a combined arrow with the same information or by a letter tile holding the letter from which the rule is derived. Because there is a junction in ε, the sequence of downward arrows cannot continue indefinitely and a letter tile as required must be situated somewhere between α and ε. Notice that such a letter tile is necessarily of type 2 and forces a combined arrow directed towards it from the left. For this reason it cannot be placed at some position like δ between the saturated blocks of the m-level lattice letters. Trying to place the letter tile at some position like γ also leads to a contradiction. For then, following the sequence of combined arrows leading to it from, say, the left, we either reach a tile ζ containing an interior line, or we meet before reaching ζ a junction tile and try to penetrate a saturated block belonging to some m-level lattice tile. In each case we get a contradiction. Hence we see that the only possible place for the letter tile is indeed β.

Reasoning similarly for the other junctions, we get a picture like in Fig. 19.

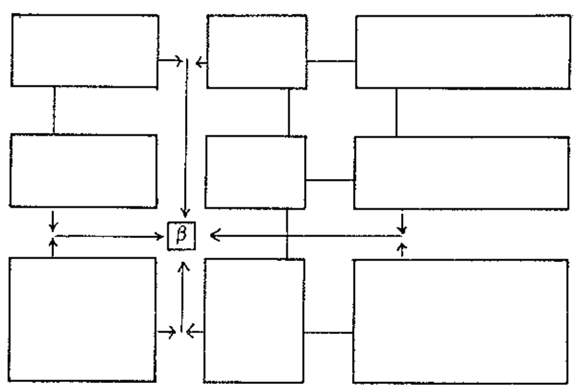

Fig. 19.

The letter tile β we obtained belongs to the $(m+1)$-level lattice, which is indeed a lattice by the synchronization of the m-level lattice tiles. Notice that the counter information along the combined arrows is determined and it crosses the required number of inner lines. Also, each of the four combined arrows intersects some exterior lines which have not already been intersected and the intersections

force "A", "B", "L", "R" tiles along these lines. We have now completed the proof of the induction step.

For the sequel, it is important to note at this point that, from the above discussion, it follows that looking at a letter tile belonging to the m-level lattice we can follow the sequence of derivations from it down to the 1-level lattice and we see a legal derivation tree of the substitution system $(\mathscr{A}, \mathscr{P})$. Furthermore, it is important to note that looking at a 2×2 quadruple of letters at the m-level lattice, and looking at the four adjacent corners of the four rules derived from them in the $(m-1)$-level lattice, the quadruple appearing in the $(m-1)$-level tiles at the corners is the right quadruple from which they were derived.

§4. The relation of $(\mathscr{T}, \mathscr{N})$ to $(\mathscr{A}, \mathscr{P})$

Theorem 4.1. *Let $(\mathscr{A}, \mathscr{P})$ be a substitution system satisfying:*

(1) *Every derivation rule*

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

satisfies $k \geqq 2$, $l \geqq 2$.

(2) *$(\mathscr{A}, \mathscr{P})$ has property A.*

Let $(\Omega, \mathbf{Z}^2)$ be the dynamical system defined by $(\mathscr{A}, \mathscr{P})$.

Let $(\mathscr{T}, \mathscr{N})$ be the tiling system described above for $(\mathscr{A}, \mathscr{P})$ and $(\Lambda, \mathbf{Z}^2)$ the dynamical system defined by $(\mathscr{T}, \mathscr{N})$; that is, Λ is the collection of all possible tilings by $(\mathscr{T}, \mathscr{N})$.

Define a map $\varphi : \Lambda \to \mathscr{A}^{\mathbf{Z}^2}$ by matching to each tiling $\lambda \in \Lambda$ the two-dimensional lattice of the letters written in the 1-level lattice.

Then φ is a map from Λ to Ω.

Proof. Let $\lambda \in \Lambda$ be a legal tiling of the plane by the system $(\mathscr{T}, \mathscr{N})$. Denote $\varphi(\lambda) = (y_{ij})_{i,j\in\mathbf{Z}}$, $\forall i, j \in \mathbf{Z}$, $y_{ij} \in \mathscr{A}$. It suffices to show that for every block $Y = Y^{(1)} = (y_{ij})$, $u \leqq i \leqq v$, $r \leqq j \leqq s$, $u, v, r, s \in \mathbf{Z}$ there exists a legal derivation tree of the substitution system $(\mathscr{A}, \mathscr{P})$ that derives a block containing $Y^{(1)}$ as a subblock.

Look at the tiles of the 1-level lattice corresponding to $Y^{(1)}$. Each of them is part of some derivation rule. The tiles of the 2-level lattice from which the tiles of $Y^{(1)}$ are derived form a rectangular array. Denote this array by $Y^{(2)}$. Notice that the derivation from $Y^{(2)}$ as described in the tiling λ is a legal derivation of the substitution system $(\mathscr{A}, \mathscr{P})$ and the block derived contains $Y^{(1)}$. Each of the tiles corresponding to $Y^{(2)}$ belongs, in the tiling λ, to some derivation rule. Let $Y^{(3)}$ be the rectangular array of tiles from the 3-level lattice from which these rules are derived. We get in this way a sequence of rectangular blocks

$Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}, \ldots$ such that from each block $Y^{(n)}$ there is a sequence of legal derivations in $(\mathscr{A}, \mathscr{P})$ of length $n-1$ at the end of which we get a block Z_n containing $Y^{(1)}$.

Lemma. *If the width (height) of the block $Y^{(i)}$ is $\geqq 3$ then the width (height) of the block $Y^{(i+1)}$ is strictly smaller than that of $Y^{(i)}$.*

Proof. Define a mapping from $Y^{(i)}$ to $Y^{(i+1)}$ by mapping each letter in $Y^{(i)}$ to the letter from which it is derived in $Y^{(i+1)}$. By the definition of $Y^{(i+1)}$ this mapping is onto. Furthermore, this mapping maps the letters in a row (column) of $Y^{(i)}$ to letters in a row (column) of $Y^{(i+1)}$. Hence the width (height) of $Y^{(i+1)}$ is smaller than or equal to the width (height) of $Y^{(i)}$.

In the case that the width of $Y^{(i)}$ is $\geqq 3$, there are three adjacent letters $y^{(i)}_{11}$, $y^{(i)}_{12}$, $y^{(i)}_{13}$ in a row of $Y^{(i)}$. Since all the derivation rules are of width $\geqq 2$ it follows that the derivation rule from which $y^{(i)}_{12}$ is derived, derives $y^{(i)}_{11}$ or $y^{(i)}_{13}$ too. Hence the mapping is not one to one and the width of $Y^{(i+1)}$ is strictly smaller than that of $Y^{(i)}$.

The proof of the assertion for height is analogous. ∎

We return to the proof of Theorem 4.1.

By the lemma, we see that the sizes of the blocks in the sequence $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}, \ldots$ are strictly decreasing until they reach one of the sizes: 1×1, 2×1, 1×2 or 2×2.

If we reach $Y^{(n)}$ such that its size is 1×1, that is $Y^{(n)}$ is a single letter, then by the above there is a legal derivation of length $n-1$ from $Y^{(n)}$ deriving a block containing $Y^{(1)}$ as needed.

Supose the minimal size is 1×2, 2×1 or 2×2. Denote it by $d\times d'$ and let $Y^{(n)}$ be a block in the sequence of this size. We have that $Y^{(n+1)}$ is of this size too. From this it is easily seen that all the $d\cdot d'$ tiles in $Y^{(n)}$ are border tiles in the derivation rules from $Y^{(n+1)}$ to $Y^{(n)}$. Remember that at corner tiles a quadruple, q, also appears from $Q=Q(\mathscr{A}, \mathscr{P})$ and the same quadruple appears in the $d\cdot d'$ neighbouring corner tiles of the $d\cdot d'$ derivation rules deriving $Y^{(n)}$ from $Y^{(n+1)}$. (In the case that $d\cdot d'=2$ there are two possible such quadruples. If $d\cdot d'=4$ only one is possible.) Notice now that this quadruple q is either $Y^{(n+1)}$ if $d\cdot d'=4$ or is $Y^{(n+1)}$ with two neighbours of it in the $n+1$ level lattice if $d\cdot d'=2$, and we have a sequence of legal derivations of length n starting from q deriving a block containing $Y^{(1)}$. Since every quadruple in Q, in a particular q, appears in a block derived from some single letter, we are in the situation of property A. Since $(\mathscr{A}, \mathscr{P})$ has, by assumption, property A, there is a legal derivation tree deriving a block containing $Y^{(1)}$. ∎

Theorem 4.2. *Let $(\mathscr{A}, \mathscr{P})$ be a two-dimensional substitution system and $(\Omega, \mathbf{Z}^2)$ the dynamical system defined by it. For every element $x=(x_{ij})_{i,j\in\mathbf{Z}}\in\Omega$ there exist $y\in\Omega$, $y=(y_{ij})_{i,j\in\mathbf{Z}}$, such that x may be derived from y by replacing each y_{ij} by some derivation rule belonging to y_{ij} in $\mathscr{P}$ and the derivation is a legal one.*

Proof. We shall assign to each letter x_{ij}, $i,j\in\mathbf{Z}$ in x a derivation rule deriving some rectangle containing it. The assignment will be made in a consistent way, the element $y=(y_{ij})\in\mathscr{A}^{\mathbf{Z}^2}$ consisting of the letters deriving each rectangle belongs to Ω and the derivations form a legal derivation of x from y.

Denote by $X^N=(x_{ij})_{|i|\leqq N,|j|\leqq N}$, $N\geqq 1$ the square of size $2N+1\times 2N+1$ centered at the origin.

Order $\mathbf{Z}^2$ in a sequence $\{(m_i, n_i)\mid i=0,1,\ldots\}$.

Since $x\in\Omega$, for every $N\geqq 1$ there exists some derivation tree T_N of the system $(\mathscr{A}, \mathscr{P})$ deriving a block containing X^N.

Define $S^0=\{T_N\mid N\geqq 1\}$.

Now, look at $x_{(m_0,n_0)}$. There are infinitely many trees T_N in S^0 deriving blocks X^N containing $x_{(m_0,n_0)}$. In each such tree there is a derivation rule from which $x_{(m_0,n_0)}$ is derived. Since the set $\mathscr{P}$ of derivation rules is finite and the number of such trees is infinite, there is some derivation rule $\alpha\in\mathscr{P}$ used to derive $x_{(m_0,n_0)}$ in the same way infinitely many times.

Define $S^1\subset S^0$ to be the collection of all the trees in S^0 in which $x_{(m_0,n_0)}$ was derived by the rule α. Look at $x_{(m_0,n_0)}$ and all the other letters in x forming the block derived by this rule α and assign to each of them the derivation rule α. Notice that in all the derivation trees in S^1 these letters are derived by this rule α.

Continue the induction. At the k-th step let $(m,n)=(m_{j_k}, n_{j_k})$ be the first pair in the sequence $\{(m_i,n_i)\mid i\geqq 0\}$ for which a derivation rule deriving $x_{(m,n)}$ has not been determined yet. Look at the trees in the infinite set S^k deriving blocks containing $x_{(m,n)}$. As above, there is some derivation rule β deriving $x_{(m,n)}$ in infinitely many such trees. Define $S^{k+1}\subset S^k$ to be the collection of all the trees in S^k in which $x_{(m,n)}$ is derived by β. Assign to $x_{(m,n)}$ and the other letters derived by β the derivation rule β. Notice that all the trees in S^{k+1} contain the derivations of all the $x_{(i,j)}$ for which derivation rules were already determined. Hence:

(1) The neighbours of $x_{(m,n)}$ were not previously grouped to some derivation rule.

(2) The derivation β of $x_{(m,n)}$ and its neighbours is consistent with the other derivations already determined such that the derivation is legal.

We proceed thus for all $k\geqq 0$ and this process defines a division of all the infinite array x into rectangular derivation rules arranged in a straight lattice of the form shown in Fig. 20.

Define $y=(y_{ij})$ to be the infinite array of the letters from which the above derivation rules were derived. It is clear that x may be legally derived from y. We have to show that $y\in\Omega$, that is, for every finite block z appearing in y there is a

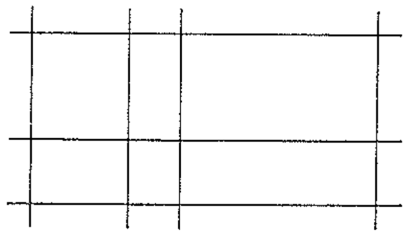

Fig. 20.

legal derivation tree of $(\mathscr{A}, \mathscr{P})$ deriving a block containing z. Look at such a block z appearing in y. From z is derived a finite block w appearing in x. There is a $k \geq 0$ such that the set $\{(m_i, n_i) \mid 0 \leq i \leq k\}$ contains all the positions of w. Looking at a tree $T_N \in S^{k+1}$ we see a tree deriving a block containing w and all the derivation rules for w are the same as those from z to w. Hence if we omit the last level of derivations from this tree, we get a legal derivation tree deriving a block containing z as required; so $y \in \Omega$. ■

Theorem 4.3. *Under the assumptions of Theorem* 4.1, *the map* φ *defined in Theorem* 4.1 *from the tilings* Λ *of the tiling system* $(\mathscr{T}, \mathscr{N})$ *to* Ω — *the dynamical system defined by* $(\mathscr{A}, \mathscr{P})$, *is onto.*

Proof. Let $X \in \Omega$. Denote $Y^{(1)} = X = (x_{ij})_{i,j \in \mathbf{Z}}$. We follow the tiling process described after the construction of $(\mathscr{T}, \mathscr{N})$. Write the letters x_{ij} in the 1-level lattice tiles. According to Theorem 4.2 there is $Y^{(2)} \in \Omega$ such that $Y^{(1)} = X$ may be legally derived from it by the rules of $(\mathscr{A}, \mathscr{P})$. Fill in each of the 1-level tiles with the rest of its information according to the derivation of $Y^{(1)}$ from $Y^{(2)}$ and complete the system of lines, arrows and junctions forced by this information as in the description of the tiling process. Notice that since $X = Y^{(1)}$ is legally derived from $Y^{(2)}$ and $Y^{(2)} \in \Omega$, the process of filling the arrows, junctions and lines may be carried out satisfying the tiling rules. In this process the tiles of the 2-level lattice are determined and filled with the letters of $Y^{(2)}$. We continue by induction. At step i we have letters of an infinite array $Y^{(i)} \in \Omega$ written in the i-level lattice. By Theorem 4.2 there is an array $Y^{(i+1)} \in \Omega$ from which $Y^{(i)}$ may be derived. We fill the information of the i-level lattice according to this derivation of $Y^{(i)}$ from $Y^{(i+1)}$. Complete the system of arrows, lines and junctions induced by it and form the $(i+1)$-level lattice tiles with the letters of $Y^{(i+1)}$ written in them. Again this process may be carried out in compliance with the rules of the tiling system since $Y^{(i)}$ is legally derived from $Y^{(i+1)}$ which belongs to Ω. (The fact that $Y^{(i+1)} \in \Omega$ ensures that the quadruple appearing in the tiles are legal ones.) We see that in this way we get a tiling $\lambda \in \Lambda$ such that $\varphi(\lambda) = X$. ■

Definition. A substitution system $(\mathscr{A}, \mathscr{P})$ will be said to have *unique derivation* if for every $x \in \Omega$ the element $y \in \Omega$ whose existence is ensured by

Theorem 4.2 is unique and there is a unique way of deriving x from y by legal derivation rules.

Let $(\mathscr{A}, \mathscr{P})$ be a two-dimensional substitution system with unique derivation. Let $x \in \Omega = \Omega(\mathscr{A}, \mathscr{P})$.

As in the proof of Theorem 4.3, there is a sequence of elements $y^{(i)} \in \Omega$, $i \geq 0$, $y^{(0)} = x$ such that $y^{(i)}$ may be derived from $y^{(i+1)}$ by legal derivation of $(\mathscr{A}, \mathscr{P})$. By the unique derivation property of $(\mathscr{A}, \mathscr{P})$ this sequence, as well as the derivations, is unique. We look at the infinite derivation tree consisting of the $y^{(i)}$, $i \geq 0$ and the derivations. Notice that this tree is either

(1) connected, or
(2) it has two components and if we look at the two parts of $x = y^{(0)}$, one in each component, they are two complementary half planes separated by a vertical or horizontal line, or
(3) it has four components and their parts in $x = y^{(0)}$ form four quadrants of the plane, separated by a vertical and a horizontal line.

We will refer to this infinite tree as the derivation tree of x.

Lemma. *Suppose* $(\mathscr{A}, \mathscr{P})$ *has unique derivation. Let* $D \subset \Omega = \Omega(\mathscr{A}, \mathscr{P})$ *be the set of all* $x \in \Omega$ *such that their derivation tree is not connected. Let* μ *be any* $\mathbf{Z}^2$ *invariant probability measure on* Ω. *Then* $\mu(D) = 0$.

Proof. By the discussion above, for every $x \in D$ there is some "separating line" separating two components of x. Let $D_h \subset D$ be the set of those elements for which there is a horizontal separating line, and $D_v \subset D$ the set of those with a vertical separating line. Clearly $D = D_v \cup D_h$. Define

$$\left.\begin{aligned} D_v^i &= \{x \in D_v \mid \text{the separating line is between columns } i \text{ and } i+1\}, \\ D_h^i &= \{x \in D_h \mid \text{the separating line is between rows } i \text{ and } i+1\}, \end{aligned}\right\} \quad i \in \mathbf{Z}.$$

Then

$$D = \bigcup_{i \in \mathbf{Z}} D_h^i \cup \bigcup_{j \in \mathbf{Z}} D_v^j.$$

It suffices to show $\mu(D_h^i) = 0$, $\mu(D_v^i) = 0$ for all $i \in \mathbf{Z}$. We prove $\mu(D_v^i) = 0$. The proof of $\mu(D_h^i) = 0$ is analogous. By the unique derivation it follows that $D_v^i \cap D_v^j = \varnothing$ for $i \neq j$. It is easily seen that $(0, k)D_v^i = D_v^{i+k}$, ($(0, k)D_v^i$ is the translation of D_v^i by the action of $(0, k) \in \mathbf{Z}^2$.) Since μ is $\mathbf{Z}^2$ invariant we have $\mu(D_v^i) = \mu(D_v^j)$ for every $i, j \in \mathbf{Z}$. In a space of finite measure, a countable collection of disjoint sets having equal measure is possible only if each of them has measure 0. We get $\mu(D_v^i) = 0$. Similarly $\mu(D_h^i) = 0$ and $\mu(D) = 0$. ■

The proof of the following lemma is completely analogous.

Lemma. *Let $(\mathcal{T}, \mathcal{N})$ be the tiling system for $(\mathcal{A}, \mathcal{P})$, $(\mathcal{A}, \mathcal{P})$ having unique derivation, and let $(\Lambda, \mathbf{Z}^2)$ be its dynamical system. Let $\Delta = \varphi^{-1}(D)$ with D as in the preceding lemma. For every $\mathbf{Z}^2$ invariant probability measure ν we have $\nu(\Delta) = 0$.* ∎

Theorem 4.4. *If the substitution system $(\mathcal{A}, \mathcal{P})$ has unique derivation and satisfies the conditions of Theorem 4.1, then the map φ defined in Theorem 4.1 is one to one almost everywhere with respect to any $\mathbf{Z}^2$ invariant measure.*

Proof. Suppose $\lambda, \lambda' \in \Lambda \setminus \Delta$ are two legal tilings and $\varphi(\lambda) = \varphi(\lambda') = X \in \Omega$. Then by the definition of φ it follows that the letter arrays written in the 1-level lattices of both tilings λ and λ' are identical to X. Let Y be the array of letters written in the 2-level lattice of λ, and Y' the array of letters written in the 2-level lattice of λ'. In the same way that we proved that the array of letters in the 1-level lattice of a tiling belongs to Ω, it may be shown that the array of letters in the k-level lattice of a tiling belongs to Ω for every $k \geqq 1$. In particular, $Y, Y' \in \Omega$. From the tiling properties it follows that X may be derived both from Y and from Y'.

By the unique derivation property of $(\mathcal{A}, \mathcal{P})$ this shows that $Y = Y'$ and that the derivation of X from $Y = Y'$ is identical in λ and in λ'. This means that the letter tiles of the 1-level lattice and the systems of lines, arrows and junctions induced by them are identical in λ and in λ'. Further, the letters in the 2-level lattices of both tilings are identical.

Now, as is easily seen we may proceed by induction and show in the same way that the equality of the letters in the k-level lattices in both tilings forces, by the unique derivation property, the equality of the tiles of the k-level lattices, and the system of arrows, lines and junctions generated by them as well as the equality of the letters in the $k + 1$ level lattices.

Now notice that the assumption that $\lambda, \lambda' \notin \Delta$ guarantees that each tile belongs to some saturated block belonging to some k-level lattice tile and hence the above process determines all the tiles in λ and λ'. Thus $\lambda = \lambda'$.

To check the remark that every tile belongs to some saturated block, take four letter tiles of the 1-level lattice touching its four corners or sides. For letter tiles of the 1-level lattice it is obvious. Since $\lambda, \lambda' \notin \Delta$ the derivation tree is connected, hence there is some letter tile in some k-level lattice such that all the four or two letter tiles surrounding our tile are derived from it. Therefore the tile belongs to the saturated block belonging to this letter tile. ∎

Remark. For $x \in D \subset \Omega$ the set $\varphi^{-1}(x)$ is finite: The only tiles which were not determined by the above process are those in a separating line and it is easy to see that the only possibility of filling the tiles in this separating line is some line or arrow containing the same information all along it. Or, if there are a vertical and horizontal separating lines, then there may be a letter tile in their intersection and four lines or arrows with the same information along the four "rays" from it. Hence there are a finite number of possibilities.

We would like the map $\varphi : \Lambda \to \Omega$ to be a homomorphism of dynamical systems $\varphi : (\Lambda, \mathbf{Z}^2) \to (\Omega, \mathbf{Z}^2)$. To achieve this we have to make some small changes in the tiling system:

The tiles of the modified system will be all the 2×2 quadruples of tiles of $\mathcal{T}$ satisfying the adjacency rules and such that the lower left tile of it is a letter tile of type 1. The adjacency rules of the modified system will be those it inherits from the unmodified one. For example the adjacency

1	2	5	6
4	3	8	7

is legal if [2 | 5] and [3 | 8] could be adjacent in the old tiling system.

1	2
4	3
5	6
8	7

is a legal adjacency if

4
5

and

3
6

were legal adjacencies in the old system. For simplicity we will use $(\mathcal{T}, \mathcal{N})$ again to denote the new system.

The correspondence of the tilings of the modified and the unmodified tiling systems is clear.

Theorem 4.5. *Let $(\mathcal{A}, \mathcal{P})$ be a two-dimensional substitution system satisfying:*

(1) *All the rules in $\mathcal{P}$,*

$$a \to \begin{matrix} x_{k1} & \cdots & x_{kl} \\ \vdots & & \vdots \\ x_{11} & \cdots & x_{1l} \end{matrix}$$

have $k \geqq 2$ and $l \geqq 2$.

(2) *$(\mathcal{A}, \mathcal{P})$ has property A.*

Let $(\Omega, \mathbf{Z}^2)$ be the dynamical system defined by $(\mathcal{A}, \mathcal{P})$. Then there is a tiling system $(\mathcal{T}, \mathcal{N})$ such that, denoting by $(\Lambda, \mathbf{Z}^2)$ the dynamical system defined by $(\mathcal{T}, \mathcal{N})$, there exists an epimorphism of the dynamical systems $\varphi : (\Lambda, \mathbf{Z}^2) \to (\Omega, \mathbf{Z}^2)$. Further, if the substitution system $(\mathcal{A}, \mathcal{P})$ has unique derivation, then φ is *an a.e. isomorphism of the dynamical systems (almost everywhere for any invariant probability measure).*

Proof. This theorem is an immediate corollary of Theorems 4.1, 4.3, 4.4 and the changes made in the tiling before this theorem. ∎

§5. Further refinement of the tiling system

We have shown that for a two-dimensional substitution system $(\mathscr{A}, \mathscr{P})$ satisfying the conditions of Theorem 4.5 having unique derivation, there is a tiling system $(\mathscr{T}, \mathscr{N})$ such that the two dynamical systems defined by them are measure theoretic isomorphic. Thus the tilings of the tiling system enjoy the same measure theoretic properties of the corresponding dynamical system defined by the substitution system.

We want to show that for substitution systems satisfying additional conditions, we obtain a tiling system such that the corresponding dynamical system reflects certain properties of the dynamical system of the original substitution system. The properties we wish the tiling to reflect are: (1) minimality, and (2) topological weak mixing.

To achieve this we suppose that the substitution system $(\mathscr{A}, \mathscr{P})$ satisfies also the following conditions:

(1) $(\mathscr{A}, \mathscr{P})$ satisfies the conditions of Theorem 4.5.
(2) $(\mathscr{A}, \mathscr{P})$ has unique derivation.
(3) $(\mathscr{A}, \mathscr{P})$ is deterministic.
(4) There are four fixed letters of $\mathscr{A}$ (not necessarily distinct) denoted by $c_{00}, c_{01}, c_{10}, c_{11} \in \mathscr{A}$ such that in every derivation rule of $(\mathscr{A}, \mathscr{P})$, c_{00} is the lower left corner, c_{10} is the upper left corner, c_{01} is the lower right corner and c_{11} is the upper right corner. Such a system is said to have fixed corners.
(5) Denote by D_v the set of elements from Ω whose "infinite derivation tree" is not connected and separated by a vertical line. D_h is the set of those elements with a horizontal separating line. We require that the following condition is satisfied: For all $x, y \in D_v$ every pair of adjacent letters appearing in x separated by the vertical separating line appears also in y separated by the vertical separating line of y. The analogous condition is imposed on the elements of D_h and pairs of adjacent letters separated by the horizontal separating line.
(6) $(\mathscr{A}, \mathscr{P})$ defines a minimal dynamical system $(\Omega, \mathbf{Z}^2)$.

A substitution system satisfying these conditions will be called *of type R*.

In order to achieve our goal we make some modifications in the laws of the tiling system $(\mathscr{T}, \mathscr{N})$ concerning the combinations of lines, arrows and "A", "B", "L", "R" in a tile. The notation $(\mathscr{T}, \mathscr{N})$ refers to the old tiling system; $(\Lambda, \mathbf{Z}^2)$ is its dynamical system. $(\mathscr{T}', \mathscr{N}')$ refers to the modified tiling system. $(\Lambda', \mathbf{Z}^2)$ is its dynamical system.

The changes are as follows:

(1) We look at all the tilings in Λ. In every such tiling $\lambda \in \Lambda$ we look at all the saturated blocks of some letter tile in some m-level lattice. In these saturated blocks we see tiles containing combinations of arrow lines and "A", "B", "L", "R". Define the set of all those tiles to be the only allowed such combinations in $(\mathscr{T}', \mathscr{N}')$.

(2) Look again at all the saturated blocks in elements of Λ and let K be the set of all the 3×3 squares appearing in them. In the new tiling system we require that any 3×3 square will be one of the squares in K.

This defines the new tiling system $(\mathscr{T}', \mathscr{N}')$.

We claim that all the previous theorems concerning $(\mathscr{T}, \mathscr{N})$ hold for $(\mathscr{T}', \mathscr{N}')$. The only non-trivial fact is that the mapping $\varphi' : \Lambda' \to \Omega$ is onto. Notice that $\Lambda' \subseteq \Lambda$, $\varphi' = \varphi |_{\Lambda'}$. Since $\varphi : \Lambda \to \Omega$ is onto the set, $P = \varphi^{-1}(x) \subset \Lambda$ is not empty. We have to show that $P \cap \Lambda' \neq \varnothing$. Suppose on the contrary that $P \cap \Lambda' = \varnothing$. Then in every $\lambda \in P$ there is some tile or group of 9 tiles contradicting the two new laws (1) and (2) above. Since P is finite there is some finite rectangular block α containing a position of such an illegal configuration from every element of P. Look at the derivation described in α, in any element of P (by the unique derivation it is the same for every $\lambda \in P$), as in the proof of Theorem 4.1, we get that this derivation is part of some derivation tree of depth m of the substitution system $(\mathscr{A}, \mathscr{P})$. Let z be the letter from which this tree was derived. Take some tiling $\mu \in \Lambda$. Look at the m-level lattice of letters of it. It describes an element of Ω. Necessarily the letter z appears in it (this follows from the minimality). Look at the derivation tree of depth m starting from z in μ. By the determinism of the substitution system $(\mathscr{A}, \mathscr{P})$, this is the same tree as above. In particular it contains the derivation described in α. Now notice that x may be such a problematic element only if it belongs to the set $D \subset \Omega$ and the separating line, or lines, pass through the block α. Furthermore, the elements of $P = \varphi^{-1}(x)$ differ only in tiles in the separating line or lines and the illegal tiles may appear only in this column and row.

Return to the proof of Theorem 4.3 and take the tiling η constructed there for x. By our assumption there is a separating row and column (or just one of them) in η. The row and column pass through the block α. Look at the block in the tiling μ colliding with α. The only difference between the two blocks is in the separating row and column where the block in μ contains some additional information. Add this information to the block α. It is seen that we can continue it along the rest of the row and column. We get some element of P in which all the block α is legal for the tiling system $(\mathcal{T}', \mathcal{N}')$ and hence there is a contradiction. We have proved:

Theorem 4.3′. *If $(\mathcal{A}, \mathcal{P})$ is of type R then the mapping $\varphi' : \Lambda' \to \Omega$ is onto.* ∎

The main tool in showing the minimality and weak mixing of the system $(\Lambda', \mathbf{Z}^2)$ is the following:

Lemma 5.1. *Let $\varnothing \neq U \subset \Lambda$ be a nonempty open set. Then U contains some cylindrical open set determined by some saturated block belonging to some m-level lattice letter.*

Proof. It suffices to prove the lemma for cylindrical sets U_β determined by some rectangular block β. Fix some $x \in U_\beta$. Look at the derivation described in x. There are several possibilities:

(1) There is a letter tile z in some m level lattice from which β was derived.

(2) There are two separating lines, a row and a column, in x and by enlarging the block β we may assume they pass through it.

(3) There is only one separating line (a row or a column) in x and it passes through β.

In case (1), we clearly get the saturated block we need by looking at the block derived from z. Suppose case 2 holds. Then by further enlarging the block β we get a block α that is derived from the quadruple

$$\begin{matrix} c_{01} & c_{00} \\ c_{11} & c_{10} \end{matrix}$$

and looks like Fig. 21. To see that the situation is indeed as in the picture, observe that looking at the m-level lattice there is a quadruple $\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}$ such that the separating row and column pass between the letters in it. Since lines between two adjacent letters of the m-level lattice crossing the separating row or column must be exterior lines, the four letters must be corners of the rules deriving them and by the fixed corners property of $(\mathcal{A}, \mathcal{P})$ we get a situation like in the picture.

Look at the 3×3 square centered around 1 (in Fig. 21), the intersection of the separating row and column. By the laws of the tiling system $(\mathcal{T}', \mathcal{N}')$ it appears in some saturated block derived from some letter "a". Let the depth of the derivation tree described in this block be l. Let the depth of the derivation trees appearing in each of the four blocks derived from the c_{ij}, $i, j = 0, 1$ in the above picture be m. It is now readily seen, by the determinism and property A, that if we take some tiling $v \in \Lambda'$ and look at a letter tile belonging to the $(m + l - 1)$-level lattice containing "a" and then at the saturated block derived from it, we see in

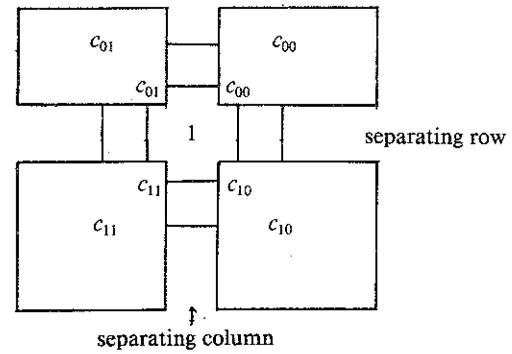

Fig. 21.

this block our block α. Notice that there is a letter tile containing "a" in the $(m + l - 1)$-level lattice because the array of letters in the $(m + l - 1)$-level lattice is in Ω and $(\Omega, \mathbf{Z}^2)$ is minimal. We can now translate the tiling v so that the block α in it will be similarly positioned and we get a saturated block as required. We have to deal next with case (3). Without loss of generality we may assume that the separating line is a column. Since the separating column passes through the block β we have a pair of horizontally adjacent letters in some m-level lattice separated by the separating column such that the block derived from this pair of tiles (including the part from the separating column) contains the block β. Notice that we used the assumption that there is no separating row. Denote these two letters by "a", "b". They are derived from a pair of letter tiles "c", "d" in the $(m + 1)$-level lattice. Look at the $(m + 1)$-level lattice. It is easy to see that the "infinite derivation tree" of the element of Ω written in it is not connected and has a vertical separating line passing between the pair "c", "d" defined above. Looking at the element of Ω written in the 2-level lattice we see that its infinite derivation tree is not connected, and has a vertical separating line passing between the two columns separated in x by the separating column. $(\mathcal{A}, \mathcal{P})$ is of type R, hence by condition (5) we conclude that the pair "c", "d" appears somewhere in the 2-level lattice of x separated by the separating column. By the determinism of the substitution system $(\mathcal{A}, \mathcal{P})$, looking at the two blocks derived from the "c", "d" of the 2-level lattice, we see in the 1-level lattice a pair of adjacent letter tiles μ, v containing "a", "b", and these two tiles are linked by a horizontal exterior line containing the same information as the line linking the pair "a", "b" of the m-level. We deduce that since the "vertical" information along the separating column is constant, the intersection ζ of the line between μ and v with the column is the same as the intersection of the line linking the m level "a", "b". Looking at the 3×3 square centered at ζ we see Fig. 22. By the laws of $(\mathcal{T}', \mathcal{N}')$ there is some tiling $y \in \Lambda'$ such that there is a tile in its l-level lattice containing the letter "e". Looking at the saturated block derived from this tile we

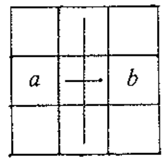

Fig. 22.

see in it the above 3×3 square. As in case (2) we see that, looking at a letter tile containing "e" in the $(m + l - 1)$-level lattice, the saturated block derived from it contains our block β. This completes the proof of the lemma. ■

Theorem 5.2. *Let $(\mathscr{A}, \mathscr{P})$ be a substitution system of type R and $(\mathscr{T}', \mathscr{N}')$ the tiling system built for it. Then the dynamical system $(\Lambda', \mathbf{Z}^2)$ defined by $(\mathscr{T}', \mathscr{N}')$ is minimal.*

Proof. Let $x \in \Lambda'$ be some legal tiling. We have to show that for every nonempty open set $U \subset \Lambda'$ there is some translation of x belonging to U. By the preceding lemma there is some saturated block belonging to a letter tile of the m-level lattice such that the cylindrical open set determined by it is contained in U. Denote by "a" the letter in this m-level tile. Look at the m-level lattice in x. The letters in it form an element of Ω. By the minimality of $(\Omega, \mathbf{Z}^2)$ we see that the letter "a" appears in it. Look at the saturated block derived from it. This is the block determining the cylindrical set and we see that some translation of x is in U. ■

Lemma 5.3. *Let $(\mathscr{A}, \mathscr{P})$ be a two-dimensional substitution system having unique derivation. For every $n \geqq 1$ there is $N \geqq n$ such that for every $m \geqq N$ if we fix some square A of size $m \times m$ of letters which is a legal block of $(\mathscr{A}, \mathscr{P})$, then in all the derivation trees of any block containing A the derivation of the $n \times n$ square B positioned at the center of A is unique. By "the derivation of the $n \times n$ square B," we mean the minimal block of letters in the last derivation step deriving the block B and the derivation rules used.*

Proof. Suppose the assertion is false. Then for some fixed $n \geqq 1$ we have a sequence of legal squares A_m of size $m \times m$, $m \to \infty$, such that for each of them there are two derivation trees deriving a block containing it and the derivation of the $n \times n$ midsquare B_m is different in each tree.

It is seen by a straightforward argument that there is an infinite array of letters $x \in \mathscr{A}^{\mathbf{Z}^2}$ such that every $k \times k$ square, $k \geqq 1$, centered around $(0, 0)$ appearing in x is a $k \times k$ midsquare of some A_m, $m = m(k)$. Clearly $x \in \Omega$. Let W denote the $n \times n$ square centered at the origin in x.

By Theorem 4.2 we find a $y \in \Omega$ such that x is derived from y. Look at the derivation of the block W in the derivation of X from y. Call this derivation γ. By

the construction of x we see that, using the notation of the proof of Theorem 4.2, for each square X^k centered at the origin, $k \geqq n$, there are two derivation trees deriving a block containing it with different derivations of W. Hence in one of these trees the derivation is different from γ. Again using the notation of 4.2, denote the last tree by T_k. Now following the proof of 4.2 we get an element $z \in \Omega$ deriving x and this derivation is necessarily different from the derivation of x from y. Hence we get a contradiction to the unique derivation. ■

Lemma 5.4. *For every saturated block B appearing in some tiling, there is a legal block A of letters such that if $\lambda \in \Lambda'$ is a legal tiling in which the block A appears in some area in the 1-level lattice, then in a specific place in λ, there appears the saturated block B.*

Proof. This lemma is a corollary to the preceding lemma. Look at the derivation tree of the saturated block B and let

$$U^0 \to U^1 \to U^2 \to \cdots \to U_{l-1} \to U^l$$

be the derivation described in it. U^0 is a single letter, and U^i are the letters in the $(l + 1 - i)$-level. By the preceding lemma there is some $M \geqq 1$ such that a legal block of size M determines the derivation of any block of size smaller than or equal to that of U^l in its center.

Now by property A of the substitution system, we can embed our derivation tree in a larger derivation tree such that if the derivation described by the latter is

$$V^0 \to V^1 \to \cdots \to V^p$$

then U^i is placed in the interior of V^{p-l+i} and is surrounded by a block of size $\geqq M$. Moreover, since the derivation rules of $\mathscr{P}$ have bounded size, we can take V^p so big that for any derivation of V^p the sizes of the last l blocks in the derivation are big enough to ensure that we see there the derivation $U^0 \to U^1 \to \cdots \to U^l$. We conclude that in every derivation of V^p the derivation of the U^i, $i = 0, 1, \ldots, l$ is as in the saturated block B. Hence if a tiling $\lambda \in \Lambda'$ has the block V^p appearing somewhere in its 1-level lattice, it contains the saturated block B written in the convex hull of the tiles where U^l is written. ■

Theorem 5.5. *Let $(\mathscr{A}, \mathscr{P})$ be a substitution system of type R. Suppose also that $(\Omega, \mathbf{Z}^2)$, the dynamical system defined by $(\mathscr{A}, \mathscr{P})$, is topologically weakly mixing. Then the dynamical system $(\Lambda', \mathbf{Z}^2)$ defined by the corresponding tiling system $(\mathscr{T}', \mathscr{N}')$ is topologically weakly mixing.*

Proof. Let $U_1, U_2, V_1, V_2 \subset \Lambda'$ be four open sets. We have to show the existence of some $(r, s) \in \mathbf{Z}^2$ such that, $i = 1, 2$, $(r, s)U_i \cap V_i \neq \varnothing$, where $(r, s)U_i$ means the translation of U_i by (r, s). By Lemma 5.1 it follows that it suffices to prove the theorem for U_i, V_i, $i = 1, 2$ cylindrical sets determined by saturated blocks B_i for U_i, C_i for V_i, $i=1,2$. From Lemma 5.4 it suffices to prove the theorem for cylindrical sets determined by four blocks of letters written in the 1-level lattice E_i for U_i, F_i for V_i, $i=1,2$. Notice that the blocks E_i, F_i, $i=1,2$ determine four open sets $U_i', V_i' \subset \Omega$. By assumption $(\Omega, \mathbf{Z}^2)$ is weakly mixing. Hence there exist $(r,s)\in \mathbf{Z}^2$ such that $(r,s)U_i' \cap V_i' \neq \varnothing$, $i=1,2$. Now, by the fact that $\varphi': \Lambda' \to \Omega$ is onto and commutes with the $\mathbf{Z}^2$ action (we are applying here the changes made before Theorem 4.5 to the new tiling system $(\mathcal{T}', \mathcal{N}')$), we get, observing that $U_i \supset \varphi'^{-1}(U_i')$, $V_i \supset \varphi'^{-1}(V_i')$, $i=1,2$, the result $(r,s)U_i \cap V_i \neq \varnothing$, $i=1,2$. ■

§6. Representing a one-dimensional substitution system by a two-dimensional tiling

Definition. Let $S_1=(\Sigma_1, \mathcal{P}_1)$ and $S_2=(\Sigma_2,\mathcal{P}_2)$ be two one-dimensional substitution systems, Σ_i the alphabets, $\mathcal{P}_i$ the derivation rules, $i=1,2$. Define their product which is a two-dimensional substitution system $G=S_1\times S_2$ whose alphabet is $\Sigma=\Sigma_1\times\Sigma_2$, and the collection of derivation rules is $\mathcal{P}=\mathcal{P}_1\times\mathcal{P}_2$ built as follows: For any two derivation rules:

$$a \to a_1a_2a_3\cdots a_n \text{ of } \mathcal{P}_1 \quad \text{(called horizontal rule)},$$

$$b \to b_1b_2b_3\cdots b_m \text{ of } \mathcal{P}_2 \quad \text{(called vertical rule)},$$

we define a derivation rule in $\mathcal{P}$:

$$(a,b) \to \begin{bmatrix} (a_1,b_m) & \cdots & & (a_n,b_m) \\ \vdots & \vdots & & \vdots \\ (a_1,b_2) & (a_2,b_2) & & (a_n,b_2) \\ (a_1,b_1) & (a_2,b_1) & \cdots & (a_n,b_1) \end{bmatrix}.$$

For a product substitution system defined like this, we modify slightly the laws governing a legal derivation of the system to ensure that in any legal block the letter of Σ_1 in all the letters of Σ appearing in a column of the block is fixed along the whole column, and that the letter of Σ_2 appearing in the letters of Σ in a row of the block is fixed along the row.

To achieve this we require that when blowing up a legal block for all the letters in a column, the same horizontal rule will be chosen from $(\Sigma_1,\mathcal{P}_1)$, and for all the letters in a row the same vertical rule from $(\Sigma_2,\mathcal{P}_2)$ will be chosen.

Theorem 6.1. *The two-dimensional dynamical system defined by the product substitution system* $(\Sigma,\mathcal{P})=(\Sigma_1,\mathcal{P}_1)\times(\Sigma_2,\mathcal{P}_2)$ *is isomorphic to the product of the two dynamical systems* $(\Omega_1,\mathbf{Z})$ *and* $(\Omega_2,\mathbf{Z})$ *defined by* $(\Sigma_1,\mathcal{P}_1)$ *and* $(\Sigma_2,\mathcal{P}_2)$ *respectively.*

Proof. Let $x\in\Omega$ be an infinite array of letters. Since $x\in\Omega$, every finite subblock of x has the property that in every column the Σ_1 letter in the Σ letters of the column is fixed. The same holds for rows: the Σ_2 letter in the Σ letters of a row is fixed. For this reason it is possible to define a map $\psi:\Omega\to\Sigma_1^{\mathbf{Z}}\times\Sigma_2^{\mathbf{Z}}$ by mapping $x\in\Omega$ to $\psi(x)=(\psi_1(x),\psi_2(x))\in\Sigma_1^{\mathbf{Z}}\times\Sigma_2^{\mathbf{Z}}$ where $\psi_1(x)$ is the projection on the Σ_1 coordinate along some row, $\psi_2(x)$ is the projection on the Σ_2 coordinate along some column. Clearly ψ commutes with the $\mathbf{Z}^2$ action on Ω and on $\Sigma_1^{\mathbf{Z}}\times\Sigma_2^{\mathbf{Z}}$. It is also clear that ψ is one to one. We have to show that the image of ψ is exactly $\Omega_1\times\Omega_2=\Omega$. Notice that in the derivation process of legal blocks for the two-dimensional substitution system $(\Sigma,\mathcal{P})$ we actually create the product of two one-dimensional derivation trees of equal depth, one from the system $(\Sigma_1,\mathcal{P}_1)$ and the other belonging to $(\Sigma_2,\mathcal{P}_2)$. Further, the product of two legal derivation trees of equal depth, one from each system, gives a legal derivation tree for $(\Sigma,\mathcal{P})$. This shows that the collection of legal two-dimensional blocks defined by $(\Sigma,\mathcal{P})$ corresponds to the product of the collection of legal words of $(\Sigma_1,\mathcal{P}_1)$ with the collection of legal words defined by $(\Sigma_2,\mathcal{P}_2)$. To see this, suppose u_i is a legal word defined by the system $(\Sigma_i,\mathcal{P}_i)$, $i=1,2$. Then there is a tree T_i deriving a word containing u_i. Let d_i be the depth of the tree T_i, $i=1,2$. It is possible to embed each of these trees in a tree R_i of depth $d\geqq d_1, d_2$ and then the product of the two trees R_i gives a derivation tree of $(\Sigma,\mathcal{P})$ of a block containing the product of the u_i's. It follows that ψ is a map from Ω onto $\Omega_1\times\Omega_2$. ■

A modification of the tiling

In order to represent a product substitution system of this kind by a tiling system, we use the construction of a tiling system described in the previous sections for a two-dimensional substitution system with small changes.

(1) In a horizontal exterior line we require that the vertical derivation rule from $(\Sigma_2,\mathcal{P}_2)$ in each of the two neighbouring rules is the same.

(2) In a vertical exterior line we require the horizontal derivation rule from $(\Sigma_1,\mathcal{P}_1)$ in each of the two neighbouring rules is the same rule.

The rest of the tiling system structure is the same. These modifications guarantee that the derivation trees we see written in the tiling are indeed legal derivation trees of the product substitution system $(\Sigma,\mathcal{P})=(\Sigma_1,\mathcal{P}_1)\times(\Sigma_2,\mathcal{P}_2)$.

Next we show that a product substitution system $G=S_1\times S_2$, for two one-dimensional substitution systems S_1, S_2, satisfies the conditions of Theorems 4.1 and 4.3.

Theorem 6.2. *Let* $S_1=(\Sigma_1,\mathcal{P}_1)$ *and* $S_2=(\Sigma_2,\mathcal{P}_2)$ *be two one-dimensional substitution systems satisfying the conditions*:

(1) *The length of every derivation rule of each of them is at least* 2.

(2) *There is at least one derivation rule for each letter.*
Let $G = (\Sigma, \mathscr{P}) = S_1 \times S_2$ be their product substitution system. Then the two-dimensional substitution system G satisfies:

(1) *Every derivation rule of G is of height and width $\geqq 2$.*

(2) *G has property A.*

Proof. (1) is clear from the assumption that all the one-dimensional derivation rules of S_1 and S_2 are of length $\geqq 2$ and the height and width of a derivation rule of G are the lengths of a rule from S_1 and a rule from S_2.

Property A follows from the observation that the legal derivation trees of G are products of two derivation trees of equal depth, one from the substitution system S_1 and one from S_2. Suppose that W is a legal block generated by G. Let

$$\alpha = \begin{pmatrix} (a, u) & (b, u) \\ (a, v) & (b, v) \end{pmatrix}$$

be a quadruple appearing in W. Suppose there is a legal derivation in G of a block B from α. Then there is a legal derivation tree T in G deriving a block containing W. This tree is the product of two derivation trees T_1 and T_2. T_i is a derivation tree of the substitution system S_i. Look at the derivation of B from α. We see that this derivation is the product of a pair of derivation trees R_1^a, R_1^b belonging to S_1 from a, b by a pair of derivation trees R_2^u, R_2^v belonging to S_2 with roots u, v.

Take the tree T_1 and attach at the vertex corresponding to a the tree R_1^a and at the vertex corresponding to b the tree R_1^b. At each other leaf vertex attach some derivation tree of the same depth like R_1^a; there is such a tree since there is a derivation rule for every symbol in Σ_1. We get in this way a derivation tree T_1'. In the same way we attach R_2^u, R_2^v to T_2 at the vertices corresponding to u, v and trees of equal depth at any other leaf vertex to get a derivation tree T_2'. Look now at the product derivation tree $T' = T_1' \times T_2'$. This is a legal derivation tree of the two-dimensional substitution system $G = (\Sigma, \mathscr{P})$ and it gives the derivation required in property A. ∎

Definition. A one-dimensional substitution system $S_1 = (\Sigma_1, \mathscr{P}_1)$ is said to have *unique derivation* if for every $x \in \Omega_1$ where $(\Omega_1, \mathbf{Z})$ is the dynamical system defined by S_1, there exists exactly one $y \in \Omega_1$ such that x may be derived from y by substituting every letter in y according to some derivation rule of $\mathscr{P}_1$. Further there is only one way of deriving x from y.

Theorem 6.3. *If the one-dimensional substitution systems S_1 and S_2 have unique derivation, then the product substitution system $G = S_1 \times S_2$ also has unique derivation.*

Proof. Let $(\Omega, \mathbf{Z}^2)$ be the dynamical system defined by G. Let $x \in \Omega$ and $y \in \Omega$ be such that x may be derived from y. Let $\psi = (\psi_1, \psi_1) : \Omega \to \Omega_1 \times \Omega_2$ be the isomorphism in the proof of Theorem 6.1. Denote $x_1 = \psi_1(x)$, $x_2 = \psi_2(x)$, $y_1 = \psi_1(y)$ and $y_2 = \psi_2(y)$.

Notice that by the definition of G it follows that $x_i, y_i \in \Omega_i$, $i = 1, 2$ and x_i may be derived from y_i by substituting each letter in y_i by some derivation rule of S_i. Hence by the unique derivation of S_i, y_i, $i = 1, 2$, as well as the derivation of x_i from y_i, is uniquely determined by x_i. Furthermore, the derivation of x from y is the product of the two derivations of x_1 from y_1 and of x_2 from y_2. Since these derivations are unique, we get that y is unique as well as the derivation of x from y. ∎

Theorem 6.4. *Let $S_1 = (\Sigma_1, \mathscr{P}_1)$ and $S_2 = (\Sigma_2, \mathscr{P}_2)$ be two one-dimensional substitution systems satisfying*:

(1) *The length of every derivation rule is $\geqq 2$.*

(2) *There is a derivation rule for every letter in Σ_i.*

Let $(\Omega_i, \mathbf{Z})$ be the dynamical system defined by S_i, $i = 1, 2$. Then there exists a tiling system $(\mathscr{T}, \mathscr{N})$ defining a dynamical system $(\Lambda, \mathbf{Z}^2)$ such that there is an epimorphism (of dynamical systems) $\varphi : (\Lambda, \mathbf{Z}^2) \to (\Omega_1 \times \Omega_2\ \mathbf{Z}^2)$. Furthermore, if S_1 and S_2 have unique derivation, then φ is an a.e. isomorphism.

Proof. The theorem follows from Theorems 6.1, 6.2, 6.3 and from observing that Theorem 4.5 holds also for modified tiling systems and product substitution systems. The proof of Theorem 4.5 in this form is analogous to the proof in the original form. ∎

We remark that the analogues of Theorems 5.2 and 5.5 hold for product substitution systems. This is the form of these theroems used for our examples of tilings.

Theorem 6.5. *Let $S = (\Sigma, \mathscr{D})$ be a one-dimensional substitution system, $(\Omega, \mathbf{Z})$ the dynamical system defined by it. Assume the following conditions hold*:

(1) *S is deterministic.*

(2) *The length of all the rules is $\geqq 2$.*

(3) *S has unique derivation.*

(4) *There is a derivation rule for every letter of Σ.*

(5) *There are $b, e \in \Sigma$ such that every derivation rule of S is of the form*

$$a \to b x_2 \cdots x_{k-1} e, \qquad a, x_i \in \Sigma.$$

(6) *$(\Omega, \mathbf{Z})$ is minimal.*

(7) *$(\Omega, \mathbf{Z})$ is topologically weakly mixing.*

Then the product substitution system $S \times S$ satisfies the conditions of Theorems 5.2 and 5.5. Hence the tiling system constructed for the product substitution system defines a minimal topologically weakly mixing system.

Proof. Let $\mathscr{A} = \Sigma \times \Sigma$, $\mathscr{P} = \mathscr{D} \times \mathscr{D}$ be the product substitution system. By Theorem 6.2, $(\mathscr{A}, \mathscr{P})$ has property A. Its rules have height and width $\geqq 2$. by Theorem 6.3 it has unique derivation. Clearly $(\mathscr{A}, \mathscr{P})$ is deterministic. The corners of every rule in $\mathscr{P}$ are (b, b), (b, e), (e, e), (e, b), hence $(\mathscr{A}, \mathscr{P})$ has fixed corners. Notice that two horizontally adjacent letters of $\mathscr{A}$, separated by a vertical separating line in any $x \in \Omega = \Omega(\mathscr{A}, \mathscr{P})$, are of the form (e, a), (b, a), $a \in \Sigma$. Further, by the minimality of $(\Omega, \mathbf{Z})$ in every "disconnected" $x \in \Omega$ with separating vertical line there is a horizontally adjacent pair (e, a), (b, a) separated by the vertical line for every $a \in \Sigma$. The minimality of $(\Omega, \mathbf{Z})$ clearly implies the minimality of $(\Omega \times \Omega, \mathbf{Z}^2)$. The topological weak mixing property of $(\Omega, \mathbf{Z})$ implies that $(\Omega \times \Omega, \mathbf{Z}^2)$ is weakly mixing too. We see that the product substitution system $(\mathscr{A}, \mathscr{P})$ satisfies all the conditions of Theorems 5.2 and 5.5. The proof of these theroems can be directly applied to prove the same assertions for product substitution systems. Alternatively, one can easily check that for deterministic substitution systems, the product substitution system can be treated as an ordinary two-dimensional substitution system. ■

§7. Applications and examples

In this section we will give several examples of one-dimensional substitution systems which enable us to show the existence of tiling systems with various dynamical properties.

I. The Morse System. The Morse system is given by the substitution system with alphabet $\Sigma = \{0, 1\}$ and derivation rules

$$\begin{aligned} 0 &\to 0 \quad 1 \\ 1 &\to 1 \quad 0 \end{aligned} .$$

By results of Martin ([3]) this system has unique derivation. Hence applying Theorem 6.4 there is a tiling system defining a dynamical system isomorphic to the product of the Morse system with itself.

II. A Minimal Topological Weakly Mixing System. In [4] Martin gives conditions for a deterministic one-dimensional substitution system over an alphabet with two letters assuring that the dynamical system defined by it will be minimal and topologically weakly mixing. Martin proves also that the substitution system has unique derivation. In particular he proves ([4], Example 2) that the substitution system $S = (\{0, 1\}, \mathscr{P})$ with $\mathscr{P}$:

$$\begin{aligned} 0 &\to \ 0 \quad 0 \quad 0 \quad 0 \quad 1 \\ 1 &\to 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \end{aligned}$$

defines a minimal topologically weakly mixing dynamical system. This system satisfies the conditions of Theorem 6.5 and we get a tiling system defining a minimal topological weakly mixing system.

III. The Chacon System. The Chacon system may be defined by the substitution system C (see Petersen [5], page 216):

$$\begin{aligned} 0 &\to 0\,0\,1\,0 \\ 1 &\to 1 \end{aligned} .$$

In this form this substitution system does not satisfy the conditions of Theorem 6.4. However, by certain changes in the substitution system and in the tiling system built for its product with itself we can get a tiling system representing the product of the Chacon system with itself. Notice that in the system C there never appears a pair of successive 1's, so that after every symbol 1 there appears 0. We *attach* the symbol 1 to the 0 immediately succeeding it and form a new symbol $\overline{10}$. Define also a new symbol $\bar{0}$. The new derivation rules will be:

(1) $\bar{0} \to \bar{0}\,\bar{0}\,\overline{10}$,
(2) $\overline{10} \to \overline{10}\,\bar{0}\,\overline{10}$,
(3) $\bar{0} \to 0\,0\,1\,0$,
(4) $\overline{10} \to 1\,0\,0\,1\,0$.

Denote this substitution system by C'. We look at a product substitution system $C' \times C'$. The alphabet Σ is not all the possible pairs but

$$\Sigma = \Sigma_1 \cup \Sigma_2, \quad \Sigma_1 = \{(0, 0), (0, 1), (1, 0), (1, 1)\},$$

$$\Sigma_2 = \{(\bar{0}, \bar{0}), (\bar{0}, \overline{10}), (\overline{10}, \bar{0}), (\overline{10}, \overline{10})\}.$$

This dictates that as rules of $C' \times C'$ we take only products of the rules (1), (2) with themselves and of rules (3), (4) with themselves. For this substitution system we build a tiling system such that letter tiles of type 1 contain only letters of the set Σ_1. The letter tiles of type 2 contain only letters of the set Σ_2.

Notice that looking at such a tiling from the "2-level lattice above" we see a tiling of the product substitution system of the one-dimensional substitution system C'' with itself, where C'' has alphabet $\{\bar{0}, \overline{10}\}$ and derivation rules:

$$\begin{aligned} \bar{0} &\to \bar{0}\,\bar{0}\,\overline{10} \\ \overline{10} &\to \overline{10}\,\bar{0}\,\overline{10} \end{aligned} .$$

This substitution system satisfies the conditions of Theorem 6.4. Moreover, by the results of Martin [3] it has unique derivation.

We next notice that every derivation of the system C can be modified by attaching each 1 to its successive 0, getting the symbol $\overline{10}$ and replacing each of the left 0's by $\bar{0}$. We get in this way a legal derivation of the system C''. Taking a derivation tree of C'' and performing a derivation according to rules (3) or (4) of C', we get a legal word derived by the original substitution system C. Moreover, every legal word of C may be derived in this way since, starting from a derivation tree of C, we can make the modifications described above only as far as the level from which the leaves (terminal vertices) of the tree are derived. Combining this observation with the fact that in the tiling we described we form in the 2-level lattice legal blocks of the product system $C'' \times C''$ and that from the 2-level lattice to the 1-level lattice we pass using products of rules (3) and (4) by themselves, we get that the blocks appearing in the 1-level lattice are exactly the legal block of the product system $C \times C$. Hence φ is a homomorphism from the tilings onto the product of the Chacon system with itself.

To see that this mapping is an a.e. isomorphism we use [4], page 217, Lemma 5.5, which asserts that C has unique derivation and hence that the letters in the 1-level lattice determine their positions in the derivation rules and the letters of the 2-level lattice. From the 2-level lattice upwards we have uniqueness by the unique derivation property of the substitution sysem C''.

§8. Undecidability of the tiling problem

Hao Wang raised in [7] the following problem:

Tiling problem. Given a tiling system, is there a legal tiling of the entire plane by this system? Wang conjectured that there is a procedure for deciding whether such a tiling exists. This conjecture was proved false by Berger [1].

In this section we sketch how this undecidability may be proved using the tiling built for a substitution system.

The idea of the proof is to show the undecidability of an analogous question about two-dimensional substitution systems and then to show that this question can be reduced to a tiling problem. The question whose undecidability we show is:

Derivation tree problem. Given a two-dimensional substitution system with rules of height and width $\geqq 2$, is there a derivation tree of every depth for it?

The reduction to a tiling problem. Given such a substitution system $(\mathscr{A}, \mathscr{P})$ we can effectively build a tiling system $(\mathscr{T}, \mathscr{N})$ for it as described above, omitting the information of legal quadruple in the corner tiles. It may easily be verified that the existence of legal tilings for $(\mathscr{T}, \mathscr{N})$ is equivalent to the existence of derivation trees of every depth for $(\mathscr{A}, \mathscr{P})$. Thus this reduction step is completed.

Undecidability of the derivation tree problem. The idea of the proof of this undecidability is to show that the halting problem of a Turing machine

with no input may be reduced to it. *Sketch of the reduction*: Given a Turing machine $G = \langle Q, \{0, 1\}, q_{10}, A \rangle$ we define a two-dimensional substitution system such that, if we look at any derivation sequence of it, starting from a single symbol $U^0 \rightarrow U^1 \rightarrow U^2 \rightarrow \cdots \rightarrow U^l$, then in each U^i we get some set of symbols arranged at the corners of some zigzagging line like that in Fig. 23, such that they

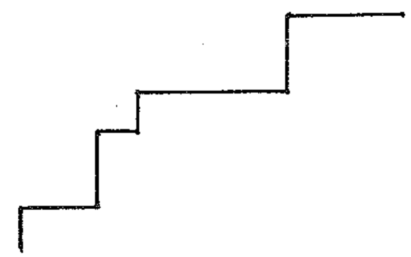

Fig. 23.

describe the state of the Turing machine and its tape after j moves. The derivation rules are designed so that in the block U^{i+1} derived from U^i we see in the symbols derived from the symbols on the zigzagging line a similar configuration describing the state of the machine after one more move, unless there is no possible move from this state, in which case there is no possible derivation from U^i. The key idea is that symbols arranged in such a zigzagging line can force a synchronization of their derivation rules used, by the requirement that the rules for symbols in one row are of the same height and symbols in one column all have the same width. The two-dimensional substitution system constructed for such a Turing machine has rules of height and width $\geqq 2$.

From the above propery of the substitution system it is seen that the existence of derivation trees of every depth implies that the Turing machine G does not halt. Further, the substitution system is constructed so that it can be shown that if the Turing machine G does not halt, there are derivation trees of all depths and the reduction is complete. Hence the tiling problem is undecidable.

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PROPAGATION OF MICROCONTINUITY FOR NONSTANDARD POLYNOMIALS

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Abstract. This paper explores the differences between real and complex microcontinuity for hyperreal polynomials, with hypernatural degree and nonstandard coefficients. On the real line, complex microcontinuity differs from real microcontinuity in replacing the coefficients with their absolute values. Apart from this feature, not much analogy is found between (absolute) convergence of series and (absolute) microcontinuity of infinite polynomials, even if these are infinite partial sums of a standard series. Real microcontinuity may be confined to isolated monads, whereas complex microcontinuity always propagates a noninfinitesimal distance. An infinite partial sum of a power series can be microcontinuous outside the circle of convergence.

1. Introduction

The set of natural, real, or complex numbers is denoted by **N**, **R** and **C** respectively, and their nonstandard extensions by ***N**, ***R** and ***C** (*hypernatural*, *hyperreal* or *hypercomplex* numbers). The external relation of being infinitely close is denoted $\approx$. (See [7].)

At its most fundamental level, continuity of a function (standard or not) means that infinitesimal fluctuations of the argument cause infinitesimal variations of the value. For continuity at a fixed point (say 0), the hyperreal monad around the point is taken into account in the case of a (hyper)real function, and the (hyper)complex monad for a (hyper)complex function. Hence the following definitions of S-*continuity*:

the internal ***R**–***R** mapping f is S-continuous at $x_0 \in {}^*\mathbf{R}$ if and only if

$$\varepsilon \in {}^*\mathbf{R} \;\&\; \varepsilon \approx 0 \Rightarrow f(x_0+\varepsilon) \approx f(x_0); \tag{1}$$

the internal ***C**–***C** mapping f is S-continuous at $x_0 \in {}^*\mathbf{C}$ if and only if

$$\eta \in {}^*\mathbf{C} \;\&\; |\eta| \approx 0 \Rightarrow |f(x_0+\eta) - f(x_0)| \approx 0. \tag{2}$$

Definition (2) can be applied to a hyperreal polynomial

$$P(x) = \sum_{j=0}^{n} a_j x^j \qquad (n \in {}^*\mathbf{N};\ a_0, a_1, \ldots, a_n \in {}^*\mathbf{R};\ x \in {}^*\mathbf{R})$$