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DUALIZATION OF MULTIGRIDS

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Résumé - Une partie substantielle de la théorie des quasi-cristaux dépend sur la dualisation des "multigrids". L'article présent veut traiter ce sujet systématiquement. En particulier on considère la question si les parallélépipèdes de la dualisation actuellement couvrent tout l'espace uniquement.

Abstract - A substantial part of quasicrystal theory depends on the dualization of multigrids. The paper attempts a systematic treatment, and in particular discusses the question of global non-overlapping.

1. INTRODUCTION

1.1. The main result of the author's paper [1] on Penrose patterns was that the patterns can be obtained as the duals of very simple figures, called pentagrids. The construction of the duals can at once be extended to n-dimensional situations, giving rise to an abundance of quite regular (but in most cases non-periodic) space coverings by means of parallelotopes (n-dimensional parallelograms).

1.2. In a recent paper [2] an extensive study was made about multigrids and their Fourier transform. The present paper reports about some of this work, generalizing and specializing it occasionally. In one respect the present paper goes beyond [2]: we investigate the question of whether the parallelotopes generated by the multigrid method actually cover the whole space without overlap. It will be explained that the multigrid dualization has a continuous analog where the non-overlapping is guaranteed by a theorem of Hadamard about differentiable maps with a non-vanishing Jacobian.

1.3. The paper tries to be systematic, starting on a general basis, and introducing specializations only when needed. The author thinks that matters can be obscured by bringing in things like orthogonality and symmetry at an early stage.

So much has been written about the dualization of grids that it is impossible to avoid duplication. In particular the reader will find that the presentation of section 3 is very close to the one of Gähler and Rhyner in [3]. These authors also formulate the condition (3.2) and claim that it guarantees local non-overlapping in 3-space. In the particular two-dimensional case of [1] it was shown to give global non-overlapping. In the one-dimensional case there is no difference between local and global non-overlapping.

1.4. Since the subject of matching conditions cannot be easily treated in a general setting, the paper will not treat it. Nevertheless something may be said here in this introduction.

In the case treated in [1] it was shown that Penrose's simple matching condition for the two kinds of rhombuses enforce the pentagrid solutions, but for most other cases such a simple thing does not seem to hold. As a test-case F. Beenker [4] took the squares and the 45° rhombuses in the plane, and was able to show that there does not exist a matching condition that enforces the grid solutions ("matching condition" now means: a condition involving only a finite number of direct or indirect neighbors of each piece). In that sense Beenker's case was as disappointing as the one-dimensional case (see [5]). But there is hope for the three-dimensional icosahedral case (with space coverings by means of two kinds of rhombohedra, a thick one and a thin one). Some investigators seem to be quite optimistic about it (although it will be much more complicated than the Penrose case).

1.5. In connection with these thin and thick rhombohedra we draw attention to a publication by Miyazaki and Takada [6], who in a pictorial way describe space-fillings with them. They also had a picture of a piece of a Wieringa roof (see [1], section 6), which projects into a Penrose pattern.

Another early publication, where duals (but no duals of grids) were considered in order to get tilings by rhombuses, is H. Crapo's paper [7]. But no doubt the history of the idea goes much further back.

2. NOTATION

If m and n are positive integers then $M_{m,n}$ is the set of all real matrices with m rows and n columns. The set of all real column vectors with m entries is denoted as $\mathbf{R}^m$; it can be identified with $M_{m,1}$. Furthermore, $\mathbf{Z}^m$ is the subset of $\mathbf{R}^m$ consisting of all vectors with integral entries. The elements of $\mathbf{R}^n$ will sometimes be called points, sometimes vectors.

The length (euclidean norm) of a vector x is denoted by $|x|$.

The transpose of a matrix A will be written as A^T.

The inner product of vectors p and q will be written as (p,q), which is the same thing as $p^T q$.

If t is a real number, then $\lceil t \rceil$ (the "roof" of t) is the smallest integer $\geqslant t$.

3. MULTIGRIDS AND THEIR DUALS

3.1. The notion of a skeleton and its dual, as described in [1] for the plane, can be generalized at once to higher dimensions. Rather than trying to define the most general situation, we start from the particular case of a multigrid (for a somewhat more general case we refer to section 4.10).

A multigrid is the union of a finite number (we take m) grids, and a grid is a set of parallel $(n-1)$-dimensional hyperplanes in the n-dimensional space $\mathbf{R}^n$. The grid directions are given by m vectors $d_1, \ldots, d_m$ in $\mathbf{R}^n$, all $\neq 0$. A hyperplane of the j-th grid is given by a real number c: it is the set of all $z \in \mathbf{R}^n$ for which the inner product (d_j, z) equals c. The possible values of c in the j-th grid are given by a set C_j of real numbers. We make the restriction that every C_j has the form

$$\{\cdots, c(j,-1), c(j,0), c(j,1), \cdots\},$$

where $c(j,k-1) < c(j,k)$ for all integers k, and $c(j,k)/k \to 1$ if $k \to \pm\infty$.

The set of all $z \in \mathbf{R}^n$ with

$$c(j,k-1) < (d_j, z) < c(j,k)$$

can be called the k-th *slice* of the j-th grid. We next take integers $k_1, \ldots, k_m$, which can be considered as the entries of a single vector $k \in \mathbf{Z}^m$. We define the set $E(k)$ as the intersection of the k_1-st slice of the first grid, the k_2-nd slice of the second grid, ..., the k_m-th slice of the m-th grid. It sometimes happens that $E(k)$ is non-empty. Then we say that k satisfies the *mesh condition*, and that $E(k)$ is a *mesh* of the multigrid.

3.2. We next explain the notion of the topological dual of a multigrid. These duals will not be presented in the general topological form, but in the special form of parallelotope coverings of $\mathbf{R}^n$ (we use the term parallelotope as the n-dimensional version of a parallelogram; for $n = 3$ it is a parallelopiped). So we are misusing the word "dual": the topological duality relation loses its symmetry since we give a very special form to the objects on both sides of the relation: "multigrids" and "parallelotope coverings".

We start taking m vectors $v_1, \ldots, v_m$ (all in $\mathbf{R}^n$); the vector v_j will be attached to the j-th grid. If k ($k \in \mathbf{Z}^m$) satisfies the mesh condition, then to $E(k)$ we attach the sum

$$k_1 v_1 + \cdots + k_m v_m \tag{3.1}$$

This is a point of $\mathbf{R}^n$. The set of all points we get this way will become the set of vertices in our parallelotope covering, but we have to make restrictions on the d's and v's first. These are the following.

(i) The vectors $d_1, \ldots, d_m$ span $\mathbf{R}^n$.

(ii) No point of $\mathbf{R}^n$ belongs to more than n grid hyperplanes, and the intersection of any n grid hyperplanes is never more than a single point.

(iii) Whenever $i_1, \ldots, i_m$ satisfy $1 \leqslant i_1 < \cdots < i_n \leqslant m$ and $\det(d; i_1, \ldots, i_n) \neq 0$, we have

$$\det(d; i_1, \ldots, i_n) \det(v; i_1, \ldots, i_n) > 0 \quad . \tag{3.2}$$

The notation $\det(d; i_1, \ldots, i_n)$ means the determinant of the $n \times n$ matrix of which the columns are the vectors $d_{i_1}, \ldots, d_{i_n}$.

These conditions (i), (ii), (iii) guarantee that the points (3.1) form the vertices of a parallelotope covering of $\mathbf{R}^n$. The parallelotopes correspond to the points of the multigrid where n grid hyperplanes intersect. At such an intersection we have $i_1, \ldots, i_n$ (the indices of the n grids involved) and $k_{i_1}, \cdots, k_{i_n}$ (the hyperplane belonging to grid i_1 is the set of all z with $(d_{i_1}, z) = c(i_1, k_{i_1})$; similarly for $i_2, \ldots, i_n$). Now the parallelotope corresponding to that intersection point consists of all points

$$x_1 v_1 + \cdots + x_n v_n \quad ,$$

where $k_j \leqslant x < k_j + 1$ if j is one of the indices $i_1, \ldots, i_n$, but, if j differs from all these, then x_j equals the index (in the j-th grid) of the slice to which the intersection point belongs.

With this description we have provided the parallelotope with some parts of its boundary, in such a way that we need not make exceptions when we state that the conditions (i), (ii), (iii) guarantee that every point of $\mathbf{R}^n$ belongs to exactly one of the parallelotopes.

We shall not prove this here, but in section 5 we shall explain how this space covering can be obtained as the limit of a continuous analog.

4. COMMENTS ON THE DEFINITION OF DUAL

4.1. Condition (i) of section 3.2 has $m \geqslant n$ as a consequence.

4.2. Usually one specializes to periodic grids. We can describe this by taking $c(j, k) = k - \gamma_j$, where γ_j is a real number. So the hyperplanes of the j-th grid are given by $(d_j, z) + \gamma_j \in \mathbf{Z}$.

Further material about periodic grids can be found in sections 4.12 and 4.13.

4.3. Let us use the matrices D and V (both in $M_{m,n}$) to describe the vectors d_j and v_j. D is the matrix whose rows are $d_1^T, \ldots, d_m^T$, and V has rows $v_1^T, \ldots, v_m^T$.

In [2] the condition (3.2) did not play a role, but instead of it, we had the weaker condition that $V^T D$ is non-singular. This weaker condition is a consequence of the conditions (i) and (iii) of section 3 (by a theorem of Binet, the determinant of $V^T D$ is obtained by summation of the left-hand side of (3.2) over all possible combinations of $i_1, \ldots, i_n$). Non-singularity of $V^T D$ was shown (in [2]) to be a satisfactory basis for the evaluation of the Fourier transform (for the case of periodic grids). It guarantees that the points (3.1) are relatively dense in $\mathbf{R}^n$ (this means that there is a number r such that every sphere of radius r contains at least one point of the form (3.1)), but it does not guarantee that the parallelotopes cover $\mathbf{R}^n$ without overlap. It is easy to give a counterexample with $n = 2$, $m = 3$.

4.4. Condition (iii) of section 3.2 is obviously satisfied if $v_j = d_j$ for all j.

4.5. If all v_j have the same length, all our parallelotopes are rhombohedra.

4.6. It is not hard to study symmetry properties of the parallelotope pattern, on the basis of properties of the multigrids and the v's. We refer to [2], section 16.

4.7. Sometimes the condition $V^T D = I$ (I is the unit matrix in $M_{n,n}$) plays a role. In the case of a periodic grid (see 4.2) it has the consequence that the parallelotope vertex corresponding to a mesh is never far from that mesh. To be more precise: there is a constant s with the following property. If k satisfies the mesh condition, and z is any point of the mesh $E(k)$, then we have $|V^T Dz - V^T k| < s$. Here $V^T k$ is the parallelotope vertex corresponding to the mesh, and $V^T Dz = z$. We refer to [2], section 10.

4.8. In many interesting cases we have both $V = D$ and $V^T D = I$. This implies that the $d_1, \ldots, d_m$ can be obtained as follows. Embed $\mathbf{R}^n$ into $\mathbf{R}^m$, take an orthogonal set of m vectors $c_1, \ldots, c_m$ (each having length 1) in $\mathbf{R}^m$, and project that set orthogonally onto $\mathbf{R}^n$.

4.9. The construction of 4.8 is related to a classical way to obtain symmetry groups. Let G be a finite group of orthogonal transformations in $\mathbf{R}^m$, such that (i) each $H \in G$ maps the set $\{c_1, \ldots, c_m, -c_1, \ldots, -c_m\}$ into itself, and such that (ii) each $H \in G$ transforms the embedded $\mathbf{R}^n$ (see section 4.8) into itself. Then the restrictions of the H's to $\mathbf{R}^n$ form a group of orthogonal transformations in $\mathbf{R}^n$, and its elements transform the set $\{d_1, \ldots, d_m, -d_1, \ldots, -d_m\}$ into itself.

If we take $v_j = d_j$ for all j, these d's and v's lead to attractive multigrids, with a small number of different parallelotopes.

4.10. The dualization from multigrid to parallelotope coverings can be extended by taking, instead of a multigrid, an arbitrary set of hyperplanes in $\mathbf{R}^n$. We have to require that any finite line segment in $\mathbf{R}^n$ intersects at most a finite number of these hyperplanes, and that every half-line intersects infinitely many. Conditions (i), (ii), (iii) of section 3.2 can be formulated with infinitely many d's instead of just m, and then they guarantee that the dual is a non-overlapping set of parallelotopes, covering the whole space, at least as long as we are able to prove $\lim_{|x| \to \infty} |f(x)| = \infty$ for the functions by which we approximate the grid (see section 5.3).

4.11. In [1] quite some attention is paid to cases (with $n = 2$) where more than 2 lines pass through a point. In those cases we used perturbed multigrids, which can be considered as what we get by shifting the grids over infinitesimally small distances. It was very essential to include those cases there, for otherwise it would not be true that all Penrose patterns arise from multigrids.

Needless to say, perturbed multigrids can also be used in n dimensions, in order to keep parallelotope coverings in cases where sometimes more than n hyperplanes pass through a point.

4.12. In the case of periodic grids (see section 4.2) we can go into some more detail (cf. [2], section 10). We introduce the unit cube $\mathrm{Cu}(m)$, which is the set of all $x \in \mathbf{R}^m$ whose entries satisfy $0 < x_1 < 1, \ldots, 0 < x_m < 1$. For $k \in \mathbf{Z}^m$, the set $E(k)$ is the set of all $z \in \mathbf{R}^n$ with

$$k - Dz - \gamma \in \mathrm{Cu}(m) \quad . \tag{4.1}$$

For D see section 4.3; γ is the vector in $\mathbf{R}^m$ with entries $\gamma_1, \ldots, \gamma_m$. So the mesh condition $E(k) \neq \emptyset$ can be expressed as the existence of z such that (4.1) holds.

If k satisfies the mesh condition then $V^T k$ is a parallelotope vertex in the dual.

We can transform the mesh condition by means of the method of [1]. We take a matrix W in $M_{m,m-n}$ with rank $m - n$ and with $W^T D = 0$ (the columns of W span the orthogonal complement of the space spanned by the columns of D). Then the existence of z with the property (4.1) can be translated into

$$W^T(k - \gamma) \in W^T \mathrm{Cu}(m) \tag{4.2}$$

(see [2], formula (10.7)). Defining the set $P(W,\gamma)$ as the set of all $W^T y$ with $y \in \mathbf{R}^m$, $y - \gamma \in \mathrm{Cu}(m)$, we find that the mesh condition can be rewritten as

$$W^T k \in P(W,\gamma) \quad . \tag{4.3}$$

The closure of $P(W,\gamma)$ is the closed polytope which is the convex hull of the set of all points $W^T(\mu + \gamma)$, where μ runs through the set of vertices of the cube $\mathrm{Cu}(m)$.

From (4.3) we infer that the set of vertices in the dual (in [2] this was called the crystal pattern) is

$$\{V^T k \mid k \in \mathbf{Z}^m, W^T k \in P(W,\gamma)\} \quad . \tag{4.4}$$

The evaluations of the Fourier transform given by Elser ([8], [9]), Duneau and Katz ([10]), Gähler and Rhyner ([3]), as well as the one in [2] all make implicit or explicit use of (4.4).

Apart from [3] and [2], most authors restrict themselves to the case that $V = D$ and $V^T D = I$ (see section 4.8). In that case we can take W such that D and W form an orthogonal matrix N ($\in M_{m,m}$): the first n columns of it are those of V, the last $m - n$ columns are those of W.

Thus far we studied the vertices of the parallelotopes in the dual, but the idea involved in (4.3) can also be used to treat the faces of various dimensions. In order to avoid complicated notation, we explain our intention by means of an example.

Let a and b be two unit vectors; we take $a = (1\,0 \cdots 0)^T$, $b = (0\,1\,0 \cdots 0)^T$. We look for all $k \in \mathbf{Z}^m$ such that $V^T k$, $V^T(k + a)$, $V^T(k + b)$, $V^T(k + a + b)$ form a two-dimensional face of one of the parallelotopes of the dual. The set of such k is given by (4.4), if we only replace $P(W,\gamma)$ by $P_{a,b}(W,\gamma)$, defined as the set of all $W^T y$ with $y \in \mathbf{R}^m$, $y - \gamma \in \mathrm{Cu}_{a,b}(m)$. And the latter set is the set of all $x \in \mathbf{R}^m$ whose entries satisfy $x_1 = 1$, $x_2 = 1, 0 < x_3 < 1, \ldots, 0 < x_m < 1$.

If we had taken $a = (-1\,0\,0 \cdots 0)^T$ instead of the above value, this would have resulted in $x_1 = 0$ instead of $x_1 = 1$.

We can also get to more complicated questions this way. For example, the intersection of $P_{a,b}(Q,\gamma)$ and $P_{a,c}(W,\gamma)$ gives us the vertices where two of such prescribed faces meet simultaneously. In this way we get the kind of results that were described in [1], section 8. The sets of vertices we single out this way, are open to Fourier transforms if we just apply the Fourier transforms of the new polytopes.

4.13. In 4.12 we presented the duality as something describing (by means of the $V^T k$ with k satisfying (4.1)) the vertices of the polytopes, but we might also like to have formulas that give the edges, and more generally the higher dimensional faces. To this end we introduce a function Δ, defined on the real line, whose function values are subsets of the real line. If t is not an integer then $\Delta(t)$ will be the set consisting of the single point $\lceil t \rceil$. If t is an integer, then $\Delta(t)$ will be the open interval from t to $t + 1$.

By means of Δ we construct the function Ω, defined on $\mathbf{R}^m$; its values are subsets of $\mathbf{R}^m$. If $x \in \mathbf{R}^m$ has the entries $x_1, \ldots, x_m$ then $\Omega(x)$ is the Cartesian product

$$\Omega(x) = \Delta(x_1) \times \cdots \times \Delta(x_m) \quad . \tag{4.5}$$

The duality of the multigrid and the parallelotope covering can be seen as a relation between a point $z \in \mathbf{R}^n$ (we might say that it lies in the multigrid space) and a point $w \in \mathbf{R}^n$ (in what we might call the parallelotope covering space, or the crystal space). To an h-dimensional face of a mesh there corresponds an $(n - h)$-dimensional face of a parallelotope in the dual (here "0-dimensional face" means "vertex", and "n-dimensional face" means "the full interior"). And the relation we want to express is "z and w lie on corresponding faces". This duality relation can be expressed in a single formula by

$$w \in V^T \Omega(Dz + \gamma) \quad . \tag{4.6}$$

In particular, if z lies inside a mesh $E(k)$, then $Dz + \gamma$ lies in the cube $k - \mathrm{Cu}(m)$ (see (4.1)), where the value of Ω is the set consisting of the single point k, so in that case (4.5) just means $w = V^T k$.

We shall not explain (4.6) any further, but just remark that it can also be obtained by the limit procedure of section 5.3.

4.14. In section 3.1 the restriction was made that $c(j,k)/k \to 1$ if $k \to \pm\infty$. This anyway covers the cases of periodicity or almost periodicity of the C_j.

The reason why this restriction was made is connected with condition (5.2) that was imposed in order to make theorem 5.2 work. It is, however not hard to weaken (5.2). In particular, if $V = D$ things are easier: then theorem 5.2 can be proved with condition (5.2) replaced by

$$h_j(t) \to -\infty \quad (t \to -\infty), \quad h_j(t) \to \infty \quad (t \to \infty). \tag{4.7}$$

The author does not know whether (4.7) is still adequate when the condition $V = D$ is dropped.

5. A CONTINUOUS ANALOG

5.1. By $C_n^{(1)}$ we denote the set of all mappings of $\mathbf{R}^n$ into $\mathbf{R}^n$ of which all partial derivatives of the first order are continuous. An element f of $C_n^{(1)}$ is a vector with entries $f_1, \ldots, f_n$, and each f_i is a function of n real variables $x_1, \ldots, x_n$. The first order partial derivatives $\frac{\partial f_i}{\partial x_j}$ form a matrix J (belonging to $M_{n,n}$), and is called the Jacobian.

Vital for our purposes will be the following theorem, that Hadamard proved in 1906:

Theorem 5.1. Let $f \in C_n^{(1)}$ have the property that $|f(x)| \to \infty$ if $|x| \to \infty$, and that the Jacobian is everywhere non-singular. Then f maps $\mathbf{R}^n$ one-to-one onto itself, and the inverse mapping belongs again to $C_n^{(1)}$.

For proofs and extensions we refer to Parthasaraty's monograph [11]. Here we just indicate a proof that the map is onto, and a proof of the fact that f is one-to-one for the special case that the Jacobian J is everywhere positive definite (this will be the case in all multigrid applications with $V = D$).

We show that for every $b \in \mathbf{R}^n$ there is a $z \in \mathbf{R}^n$ with $f(x) = b$. Put $f(z) - b = g(z)$. Now $(g(z), g(z))$ is continuous, tends to ∞ if $|z| \to \infty$, so it has a minimum at some point y. At y the vector of the partial derivatives vanishes, so $2J^T(y)g(y) = 0$. Since $J(y)$ is non-singular, we have $g(y) = 0$.

Next we assume $x \in \mathbf{R}^n$, $p \in \mathbf{R}^n$, $p \neq 0$, $f(x) = f(x+p)$. Taking a real variable t $(0 \leqslant t \leqslant 1)$ we note that the derivative of $(f(x+tp), p)$ equals $p^T J(x+tp)p$. This is positive, so $(f(x), p) < (f(x+p), p)$, whence $f(x) \neq f(x+p)$. □

5.2. We take vectors $d_1, \ldots, d_m, v_1, \ldots, v_m$ (all in $\mathbf{R}^n$) satisfying conditions (i) and (iii) of section 3.2, and we take m real functions $h_1, \ldots, h_m$ of a single real variable, all continuously differentiable (so $h_j \in C_1^{(1)}$), with positive derivative everywhere:

$$h_j'(t) > 0 \quad (i = 1, \ldots, m; -\infty < t < \infty) \tag{5.1}$$

and with

$$\frac{h_j(t)}{t} \to 1 \quad (|t| \to \infty) \quad . \tag{5.2}$$

We define a function $f \in C_n^{(1)}$ by

$$f(z) = \sum_{j=1}^{m} h_j((d_j, z))v_j \quad . \tag{5.3}$$

The Jacobian $J(z)$ of f is easily evaluated by taking two constant vectors p and q (both in $\mathbf{R}^n$), and remarking that $p^T J(z)q$ equals the derivative (with respect to t) of the inner product $(p, f(z+tq))$, evaluated at $t = 0$. From (5.3) we get

$$p^T J(z)q = \sum_{j=1}^{m} (p, v_j) h_j'((d_j, z))(dj, q) \quad , \tag{5.4}$$

so

$$J(z) = \sum_{j=1}^{m} v_j h_j'((d_j, z)) d_j^T \quad . \tag{5.5}$$

The determinant of $J(z)$ can be expressed by means of Binet's theorem. With the abbreviation

$$e_j = h_j'((d_j, z)d_j$$

and with the notation of section 3.2

$$\det J = \sum \det(v; i_1, \ldots, i_n)\det(e; i_1, \ldots, i_n) \quad , \tag{5.6}$$

where the summation runs over all sets $i_1, \ldots, i_n$ with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant m$. We can now prove

Theorem 5.2. Let $h_1, \ldots, h_m$ be elements of $C_1^{(1)}$ satisfying (5.1) and (5.2). Let $d_1, \ldots, d_m, v_1, \ldots, v_m$ be vectors of $\mathbf{R}^n$ satisfying conditions (i) and (iii) of section 3.2. Then the mapping f (defined by (5.3)) maps $\mathbf{R}^n$ one-to-one onto itself.

Proof. From (5.1) we see that

$$\det(e; i_1, \ldots, i_n) = c \cdot \det(d; i_1, \ldots, i_n) \tag{5.7}$$

with $c > 0$. Now section 3.2 (iii) says that all terms in (5.6) are non-negative, and with the extra information of section 3.2 (i) we see that at least one term is positive. So we have $\det(J(z)) > 0$ for all $z \in \mathbf{R}^n$. We note that if $V = D$ it can even be shown that $J(z)$ is positive definite.

By theorem 5.1 it now suffices to show that $|f(x)| \to \infty$ for $|x| \to \infty$. Define $p(z)$ by

$$p(z) = f(z) - \sum_{j=1}^{m} (d_j, z) v_j = f(z) - V^T D z \quad . \tag{5.8}$$

$V^T D$ is non-singular (see section 4.3), so there is a positive constant c such that $|V^T D z| \geqslant 3c\,|z|$ for all $z \in \mathbf{R}^n$. Next we put

$$c_1 = \frac{c}{(1 + \sum_{j=1}^{m} |v_j|)} \quad .$$

From (5.2) we infer that a constant c_2 exists such that

$$|h_j((d_j, z)) - (d_j, z)| \leqslant c_2 + c_1 |z|$$

for all $z \in \mathbf{R}^n$, $j = 1, \ldots, m$. We now get

$$|f(z)| > |V^T D z| - |p(z)| > c\,|z| \quad (|z| > \frac{c_2}{c_1})$$

and hence $|f(z)| \to \infty$ if $|z| \to \infty$. ▯

5.3. We shall now explain how the dual of a multigrid can be obtained as the limit of what we have expressed in theorem 5.2. We shall restrict ourselves to the case of multigrids composed of periodic grids (see sections 4.2 and 4.12), although the general case can be treated in the same way. And theorems of the type of theorem 5.2 can be used for the treatment of cases as indicated in section 4.10.

For any natural number μ we select some function $\phi_\mu \in C_1^{(1)}$ that satisfies

$$\lceil t \rceil < \phi_\mu(t) < \lceil t + \mu^{-1} \rceil + \mu^{-1} \quad , \quad \phi_\mu'(t) > 0$$

for all real t, and such that $\phi_\mu(t) - t$ is periodic mod 1. And, assuming $\mu > 3$, we select a function $\psi_\mu \in C_1^{(1)}$, periodic mod 1, with the following properties:

$\psi_\mu(t) = 0 \quad (\mu^{-1} \leqslant t \leqslant 1 - 2\mu^{-1})$,

$\psi_\mu(t) = 1 \quad (1 - \mu^{-1} \leqslant t \leqslant 1)$,

$\psi_\mu(t)$ decreases for $0 \leqslant t \leqslant \mu^{-1}$,

$\psi_\mu(t)$ increases for $1 - 2\mu^{-1} \leqslant t \leqslant 1 - \mu^{-1}$.

We now specialize (5.3) by taking

$$f_\mu(z) = \sum_{j=1}^{m} \phi_\mu((d_j, z) + \gamma_j) v_j \tag{5.9}$$

and we define the real-valued function $g_\mu : \mathbf{R}^n \to \mathbf{R}$ by

$$g_\mu(z) = \sum_{j=1}^{m} \psi_\mu((d_j, z) + \gamma_j) \quad . \tag{5.10}$$

By theorem 5.2 f_μ has an inverse f_μ^{-1}, and we define the function $\theta_\mu : \mathbf{R}^n \to \mathbf{R}$ by

$$\theta_\mu(z) = g_\mu(f_\mu^{-1}(z)) \quad . \tag{5.11}$$

The parallelotope pattern generated by the method of section 3.1 can be considered as the limit of this function if $\mu \to \infty$. This is meant in the following way. If z lies in a vertex of a parallelotope, then $\theta_\mu(z) \to 0$. If z lies on a 1-dimensional edge, then $\theta_\mu(z) \to 1$. If z lies on a 2-dimensional face, then $\theta_\mu(z) \to 2$, etc. Finally, if z lies in the interior of a parallelotope then $\theta_\mu(z) \to n$.

This idea can be used to show that the parallelotopes of section 3 cover $\mathbf{R}^n$ uniquely (in the sense stated at the end of section 3.2).

5.4. The functions θ_μ studied in section 5.3 can be shown to be uniformly almost periodic.

We recall the definition (see Besicovitch [12]). A continuous mapping f of $\mathbf{R}^n$ into $\mathbf{R}^m$ is called *uniformly almost periodic* (abbreviated to u.a.p.) if for every positive number ϵ the set of ϵ-translation vectors is relatively dense. An ϵ-translation vector is a vector $p \in \mathbf{R}^n$ such that $|f(x+p) - f(x)| < \epsilon$ for all $x \in \mathbf{R}^n$. And a subset S of $\mathbf{R}^n$ is called relatively dense if there exists a positive number r such that for every $x \in \mathbf{R}^n$ there is an $s \in S$ with $|x - s| < r$.

The function f defined by (5.9) is not u.a.p.; we first have to subtract a linear part. Introducing F_μ by

$$F_\mu(z) = f_\mu(z) - V^T Dz \tag{5.12}$$

we can show

Theorem 5.3. F_μ and g_μ are uniformly almost periodic.
Proof. Putting

$$\eta_\mu(t) = \phi_\mu(t) - t \tag{5.13}$$

we have (cf. (5.8))

$$F_\mu(z) = \sum_{j=1}^{m} (\eta_\mu((d_j, z) + \gamma_j) + \gamma_j) v_j \quad .$$

For every j the function $\eta_\mu((d_j, z) + \gamma_j)$ is u.a.p.. It has a relatively dense set of periods (and a period is always an ϵ-translation vector): every $p \in \mathbf{R}^n$ with $(d_j, p) \in \mathbf{Z}$ a period. And a sum of finitely many u.a.p. functions is again u.a.p.. The proof for g_μ is similar. □

Theorem 5.4. For every μ, the function θ_μ (defined by (5.11)) is uniformly almost periodic.
Proof. We first show that the Jacobian of f_μ has a bounded inverse. The partial derivatives of f_μ are bounded since they can be expressed by means of the derivatives of the periodic function η_μ (see (5.13)). So it suffices to show that $\det(J(z))$ has a positive lower bound. This can be obtained from the beginning of the proof of theorem 5.2. For the number c of (5.7) we can take the n-th power of the lower bound of all derivatives of the $h'(t)$'s. In the particular case of f_μ, where (5.3) is specialized to (5.9), this lower bound is positive. We recall that ϕ_μ has a positive derivative and that $\phi_\mu(t) - t$ has period 1, so that the lower bound of ϕ_μ' can be evaluated in an interval of length 1. Hence it is positive.

Since ψ_μ' is continuous and periodic, we now infer that θ_μ has bounded derivatives, so there exists a positive number c such that

$$|\theta_\mu(z_1) - \theta_\mu(z_2)| \leqslant c\,|z_1 - z_2|$$

for all z_1, z_2 in $\mathbf{R}^n$.

Let us take any positive ϵ, and put $\delta = \epsilon/(1+c)$. We shall show that if p is a δ-translation vector for F_μ as well as for g_μ, then q, defined by $q = V^T Dp$, is an ϵ-translation vector for θ_μ.

Let $y \in \mathbf{R}^n$. We put

$$x = f_\mu^{-1}(y), \quad w = f_\mu(x+p), \quad v = y+q \quad .$$

We have, by (5.12),

$$w - v = f_\mu(x+p) - f_\mu(x) - q = F_\mu(x+p) - F_\mu(x) \quad ,$$

so $|w - v| < \delta$. Now

$$|\theta_\mu(v) - \theta_\mu(y)| \leqslant |\theta_\mu(w) - \theta_\mu(y)| + |\theta_\mu(v) - \theta_\mu(w)| \leqslant$$
$$|g_\mu(x+p) - g_\mu(x)| + c\delta < \epsilon \quad .$$

From the general theory we know that if two functions are u.a.p. then they are simultaneously u.a.p., so the set of possible p's is relatively dense. Since $V^T D$ is non-singular, we conclude that the set of q's is relatively dense.

□

5.5. The fact that F_μ and θ_μ are u.a.p. is not uniform with respect to μ. In the limit for $\mu \to \infty$, the property gets lost. We cannot claim the parallelotope pattern (see the end of section 5.3) to be u.a.p.. The notion of u.a.p. is not directly applicable to discontinuous functions, nor to discrete structures like the parallelotope patterns. It has to be modified in the sense that to every mesh of the multigrid we assign a positive number to be called its *tolerance*, and in formulating the property of the ϵ-translation vectors we only admit points arising from meshes with tolerance exceeding ϵ. We refer to [2], section 17, for details.

5.6. The strict periodicity of the grids is not essential for the multigrids being u.a.p. The end of the proof of theorem 5.3 shows that it suffices that the η_μ's are almost periodic. And if the ψ_μ's are almost periodic, we can get theorem 5.4 again.

5.7. That notion of tolerance also plays a role in the matter of approximate equality of two multigrids. If both are defined by means of the same D and V, but one with a vector γ, and one with a vector γ', then the following statement holds (see [2], section 17; some special cases in a preprint "Quasicrystals II" by J.E.S. Socolar and P.J. Steinhardt, 1985). If $\beta = \gamma - \gamma'$, and if for all $h \in Z(n)$ with $h^T D = 0$ we have $h^T \beta \in \mathbf{Z}$, then the two are approximately equal.

For the case of the pentagrid of [1] this reduces to the condition $\gamma_1 + \cdots + \gamma_m \in \mathbf{Z}$. In [1] the reason for having that condition was an entirely different one: the duals of these particular pentagrids were exactly those that can be turned into arrowed rhombus patterns.

6. FOURIER TRANSFORMS

The set of vertices of the dual of a multigrid can be turned into a generalized function, if we just put a Dirac delta function at each vertex. Let us restrict ourselves to the case of periodic grids, where the representation of section 4.12 can be used. The sum of the delta functions is $\sum \delta_y$, where y runs through the set (4.4), and it is this sum we want to have the Fourier transform of.

The case with general D and V was extensively treated in [2]. Here we shall not go into analytical fundamentals and details, but just display a few formulas related to the simple situation $V = D$, $V^T D = I$, mentioned at the end of section 4.12. Most authors who have written on the Fourier transform had that special case in mind.

In this case we can have the matrix N (see the end of section 4.12) orthogonal, which gives a substantial simplification in the fundamental theorem 11.1 of [2]. Let us take smooth functions f and h in $\mathbf{R}^{m-n}$, where f is the complex Fourier transform of h:

$$f(x) = \int e^{-2\pi i (x,y)} h(y)\, dy \quad .$$

In [2] this smoothness was taken in a particular sense of a set S with very smooth functions, but it would be good enough to require that all derivatives of f and h (of any order), even after multiplication with arbitrary polynomials, tend to zero at infinity. Then with

$$F = \sum_{k \in \mathbf{Z}^m} f(W^T k)\, \delta_{V^T k} \quad , \quad H = \sum_{k \in \mathbf{Z}^m} h(W^T k)\, \delta_{V^T k} \quad ,$$

H is the Fourier transform of F.

If we make a sequence of f's, converging to the characteristic function of the polytope $P(W, \gamma)$, we get the Fourier transform of the crystal pattern (cf. (4.3)).

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COMMENTS AFTER THE N.J. DE BRUIJN TALK :

M.V. JARIC.- In connection with your emphasis that the fundamental (unit) tiles for the penrose tiling can be congruently deformed to give alternative pairs of tiles. I would like to mention a fact which I believe is important when attempting to represent "real" quasicrystals as decorations of penrose tilings : a decoration of given unit tiles is not in general equivalent to a unique decoration of different, congruently deformed, unit tiles. In the view of this, it seems to me that our studies of quasicrystals should shift the focus from tilings to quasilattices.

I would also like to remark that the exact degeneracies of infinite singular and exceptionally singular penrose tilings, mentioned by Professor DE BRUIJN, occur in every finite portion of every penrose tiling. A "real" quasicrystal would be such a finite portion of the tiling.