From Computability to Quasicrystals

Thomas Fernique

Outline

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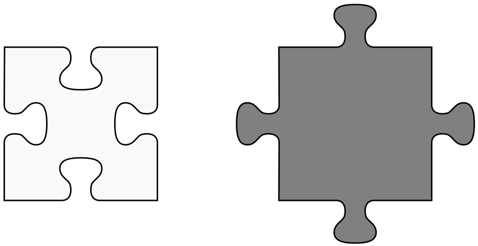
First puzzle



Two different pieces. Each can be used as many times as wanted.

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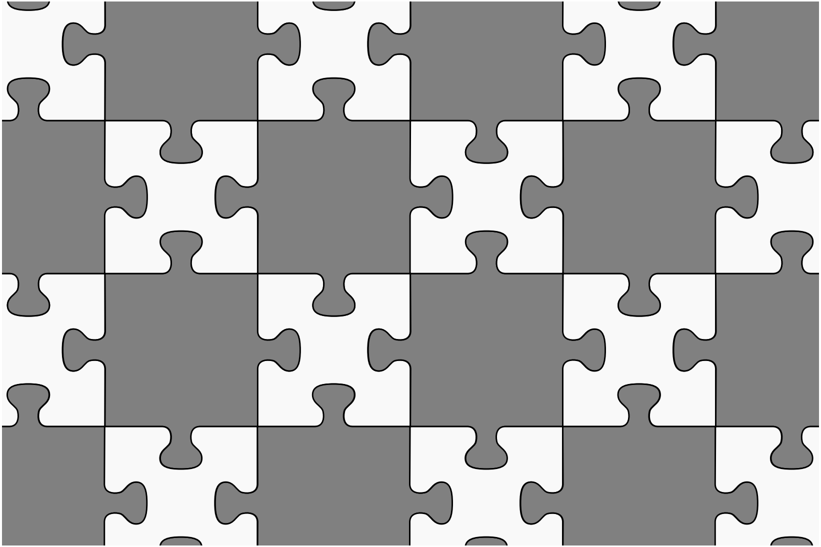

The whole plane can be tiled.

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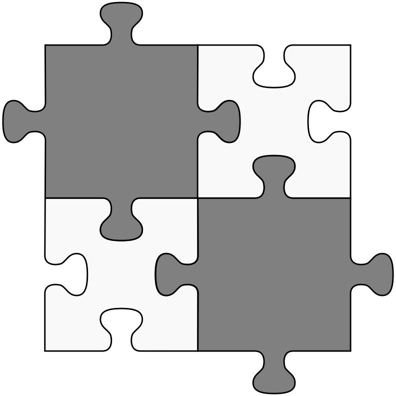
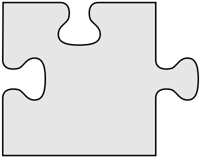


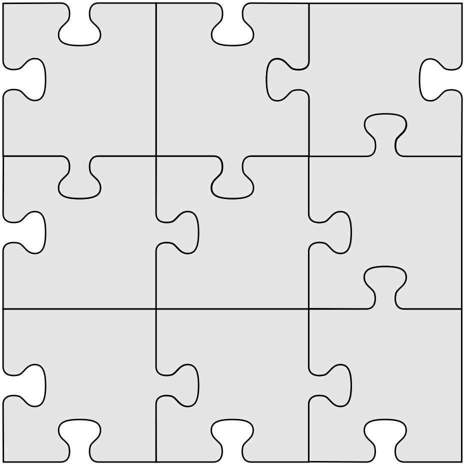
How? Just repeat this pattern *periodically*.

Second puzzle



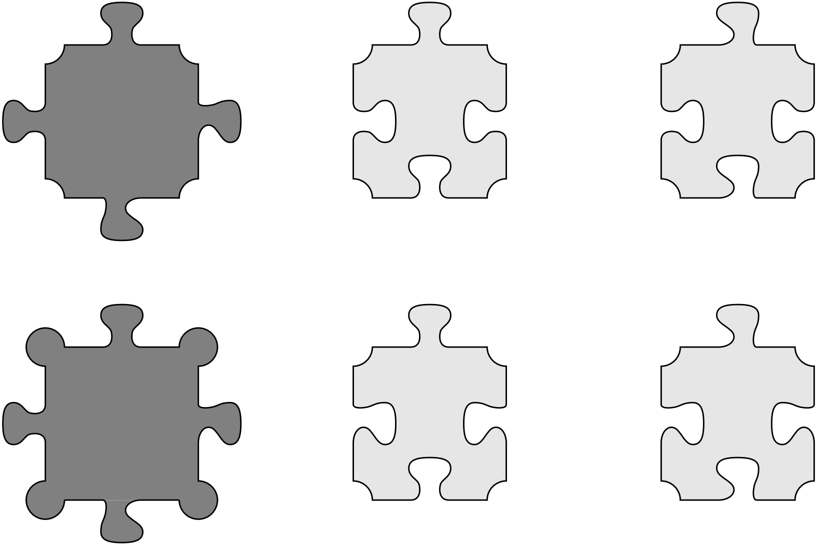
Only one piece, that can be used as many times as wanted.

Second puzzle



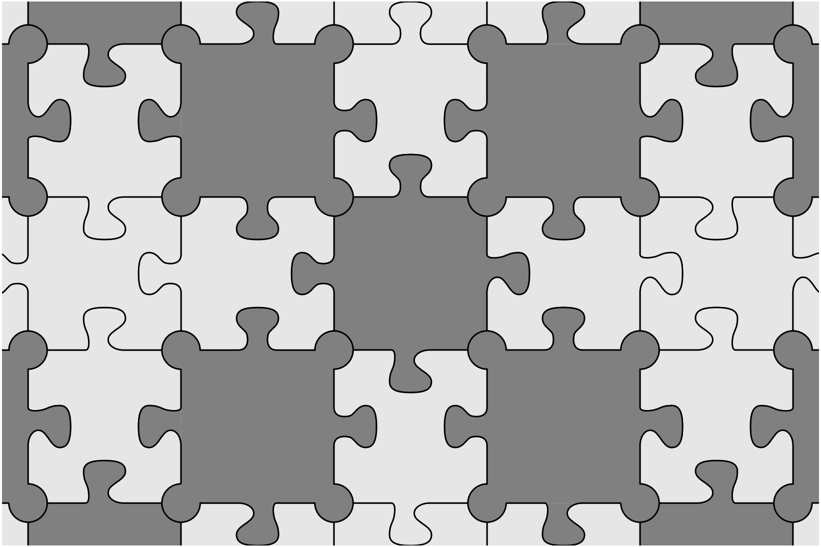
Claim: it cannot tile the plane, actually not even a 5×5 square.

Third puzzle



Six different pieces. Each can be used as many times as wanted.

Third puzzle



Theorem (Robinson, 1970): tile the plane. . . but <u>not periodically</u>!

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A Decision problem

Question: does a given finite set of pieces tile the whole plane?

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What happens for the third puzzle?

Undecidability

Theorem (Berger, 1964)

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Open question: does the existence of an aperiodic puzzle (e.g. in some restricted framework) suffice to imply undecidability?

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Puzzles

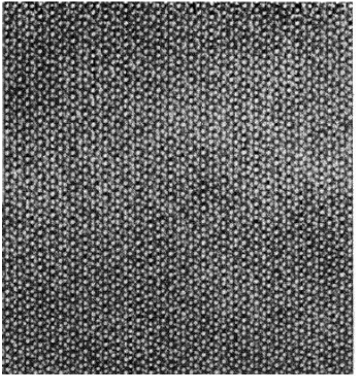
Computability

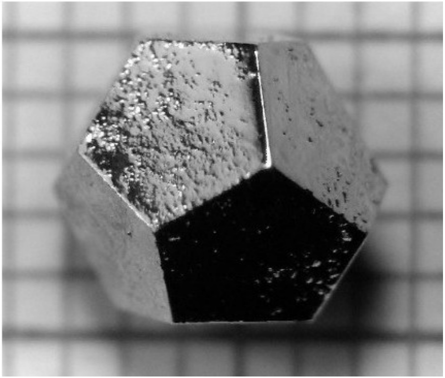
Quasicrystals

Two questions

Quasiwhat?

Periodic structure (crystal) $\Rightarrow$ long range order (e.g. facets).





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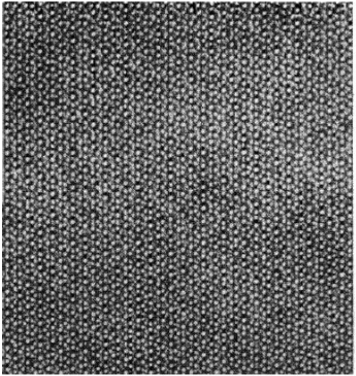

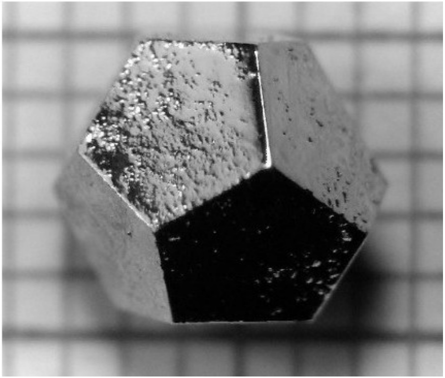

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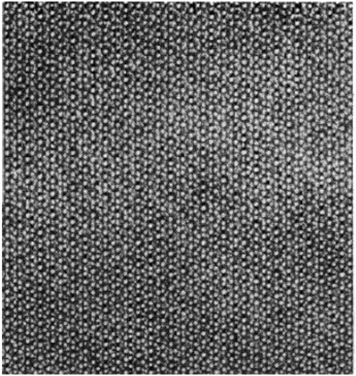

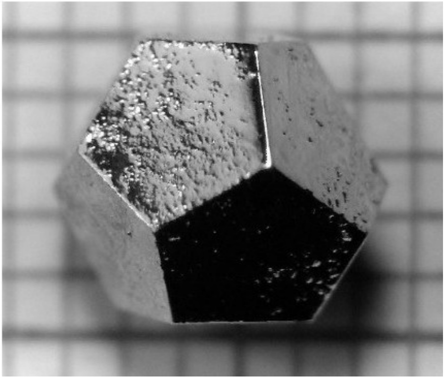

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Strong opposition (Linus Pauling: "no quasicrystal, only quasi-science") but eventually accepted (Schechtman Nobel Prize in 2011).

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Computer scientists/mathematicians: we know that since the 60's!

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Theorem (Bédaride-Fernique, 2015–2020)

Only (some) algebraic *planes are characterized by "local patterns".*

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Theorem (Fernique-Galanov, 2022)

Example of a discrete irrational plane that can be grown up to an arbitrarily small missing fraction if the initial seed is large enough.

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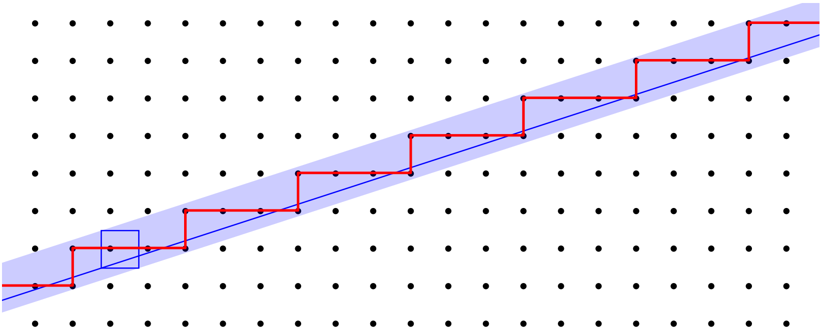
Cut and project tilings



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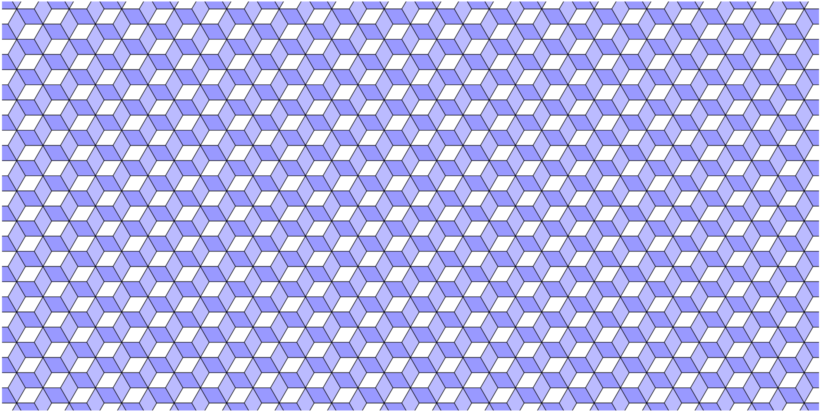

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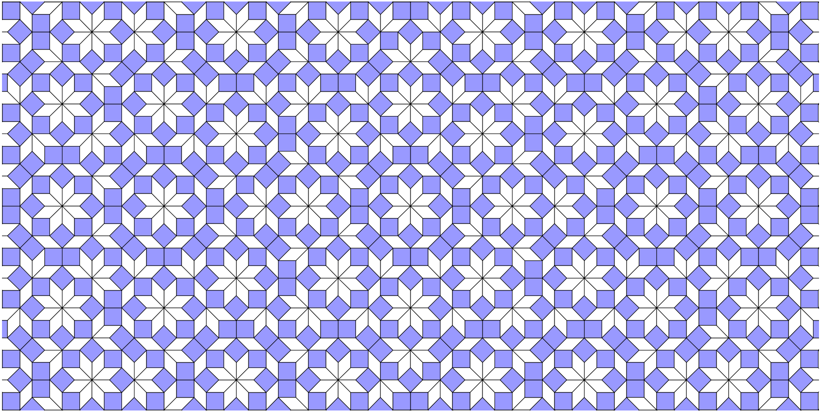

Cut and project tilings

Photo : Stan Sherer

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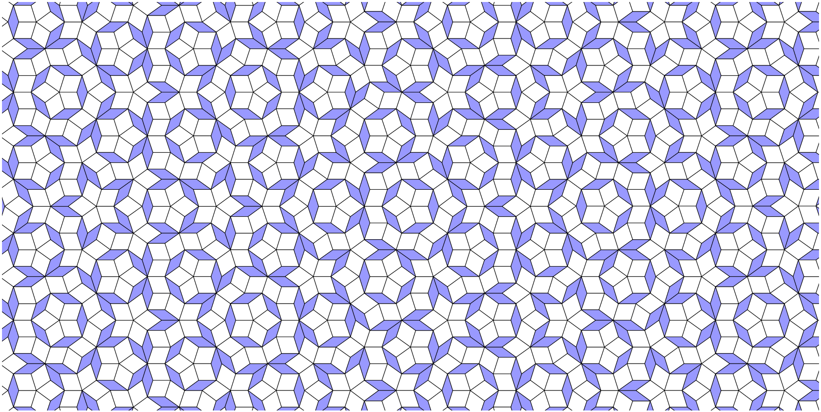
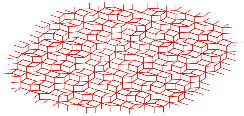


Cut and project tilings

Growth example



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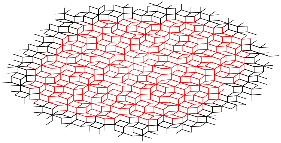

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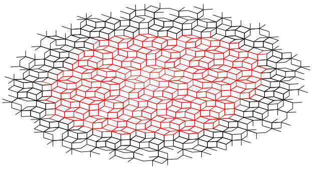

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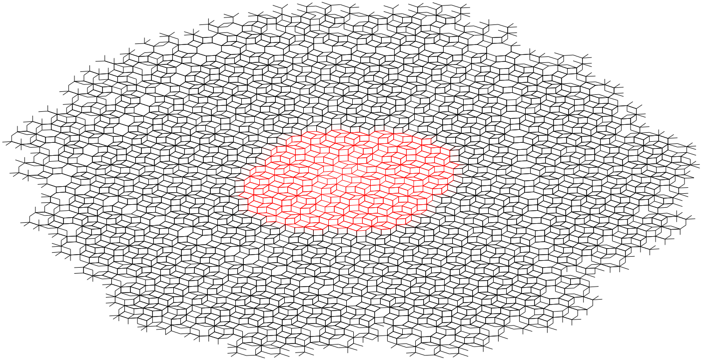

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