On the characterization of models of $\mathcal{H}^*$

Flavien BREUVART

PPS, Paris Denis Diderot; LIPN, Paris Nord

CSL-LICS, 2014

Full abstraction

Full abstraction

Denotational and operational congruences coincide :

$$[\![M]\!] = [\![N]\!] \Leftrightarrow M \equiv_o N$$

Denotational congruence $[\![M]\!] = [\![N]\!]$

Congruence induced by the denotational model :

equality of the interpertation

Operational congruence $M \equiv_o N$

Congruence induced by the language :

$$\forall C, \mathcal{O}(C(\!|M|\!)) = \mathcal{O}(C(\!|N|\!))$$

e.g., $\mathcal{O}$ = head-normalisation

Many results of full abstraction

[Abramsky & Ong 1993]

[Ouaknine 2002]

[Ehrhard *et. al.* 2014]

[Breuvart 2013]

[Hennessy & Plotkin 1979]

[Manzonetto 2009]

[Joung & Stoughton 1993]

[Hyland & Ong 2000]

[Laird *et. al.* 2011]

[Coppo,Dezani&Zacchi 1987]

[Barendregt 1984]

[Mazza & Ross 2012]

[Harmer & McCusker 1999]

[Wadsworth 1976]

[Laird 1997]

[Abramsky & McCusker 2007]

[Plotkin 1977]

[Cartwright *et.al.* 1994]

[Mazza 2009]

[Milner 1977]

[Berry *et. al.* 1985]

[Hyland 1976]

[Paolini 2003]

[Gouy 1995]

[Bucciarelli *et.al.* 2011]

[Paolini & RoncciDellaRocca 2004]

[Abramsky *et. al.* 1998]

[AJM 2000]

Our question

Previous works

[Milner1977] **Milner's theorem for PCF :**
There is a unique domain fully abstract for PCF (continuous, extentional and up to iso.)

[Gouy1995] **And for the pure λ-calculus ?** (head reduction)
There exist many non-isomorphic models fully abstract for $\mathcal{H}^*$

[Manzonetto2009] **Sufficient condition**
Any well-stratified model, *i.e.*, st

$$|f|_{k+1}(|a|_k) = |f(a)|_k \qquad |f|_0(\bot) = |f(\bot)|_0$$

is fully abstract for $\mathcal{H}^*$.

May we characterise the models fully abstract for $\mathcal{H}^*$?

Our answer

The result (informally) :

A K-model D is fully abstract for $\mathcal{H}^*$ iff D is hyperimmune,

i.e., D may contain non-well-founded chains but they are not "accessible" to λ-terms

$$\alpha_1 = a_{1,1} \to \cdots a_{1,i_1} \cdots \to a_{1,g(1)} \to \alpha'_1$$

$$\Cup$$

$$\alpha_2 = a_{2,1} \to \cdots a_{2,i_2} \cdots \to a_{2,g(2)} \to \alpha'_2$$

$$\Cup$$

$$\alpha_3 = a_{3,1} \to \cdots a_{3,i_3} \cdots \to a_{3,g(3)} \to \alpha'_3$$

$$\ddots$$

Table of contents

Models of pure λ-calculus

model of (pure) λ-calculus

Model for typed λ-calculus : Cartesian closed category $(D \Rightarrow E)$.

Model for pure λ-calculus : reflexive object in this ccc :

$$D \underset{app}{\overset{abs}{\rightleftarrows}} D \Rightarrow D$$

$$abs \circ app = id_{D \Rightarrow D}$$

Extensionality

Full Abstraction for $\mathcal{H}^*$ $\Rightarrow$ extensionality (η-equivalence) :

$$app \circ abs = id_D$$

We only consider models respecting the approximation theorem.

Definition of K-models

Extensional K-model [Krivine 1993] :

A preorder $(D, \leq)$ with "$\to$" from $\mathcal{A}_f(D) \times D$ to D s.t. :

- "$\to$" is a bijection,
- $\sqsubseteq$ is an order on $\mathcal{A}_f(D)$ defined by :
  $a \sqsubseteq b \Leftrightarrow \forall \alpha \in a, \exists \beta \in b, \alpha \leq \beta$
- $(a \to \alpha) \leq (b \to \beta) \;\Leftrightarrow\; a \sqsupseteq b \;\wedge\; \alpha \leq \beta$

Antichains

$\mathcal{A}_f(D)$ are finite antichains over D : $\forall \alpha, \beta \in a \in \mathcal{A}_f(D)$, $\alpha \not\leq \beta$

We can replace $\mathcal{A}_f(D)$ by $\mathcal{P}_f(D), \mathcal{M}_f(D)$ or intersections

- Sub-class of filter models.
- Contain historical models : D_∞, P_∞, D^*_∞...
- Contain well-stratified filter models.
- Correspond to reflexive elements of $ScottL_!$ [Ehrhard2012].

Examples (D_∞)

Scott's D_∞ :

$$D_0 = \{*\}$$
$$D_{n+1} = D_n \cup (\mathcal{A}_f(D_n) \times D_n) - \{(\emptyset, *)\}$$
$$D_\infty = \bigcup_n D_n$$

The order is uniquely generated

$$(a, \alpha) = a \to \alpha \qquad \text{if } a \neq \emptyset \text{ or } \alpha \neq *$$
$$* = \emptyset \to *$$

Examples (D_∞)

Scott's D_∞ :

$$D_0 = \{*\}$$
$$D_{n+1} = D_n \cup (\mathcal{A}_f(D_n)\times D_n) - \{(\emptyset, *)\}$$
$$D_\infty = \bigcup_n D_n$$

$(a,\alpha) = a \to \alpha$ if $a \neq \emptyset$ or $\alpha \neq *$

$* = \emptyset \to *$

The order is uniquely generated

Completion

This is an extensional completion : $D_\infty = \overline{D_0}$

D_∞ is fully abstract for $\mathcal{H}^*$
[Hyland 1976, Wadsworth 1976]

Examples (P_∞)

Park's P_∞ :

$$P_0 = \{*\}$$
$$P_{n+1} = P_n \cup (\mathcal{A}_f(P_n) \times P_n) - \{(\{*\}, *)\}$$
$$P_\infty = \bigcup_n P_n$$

$$(a, \alpha) = a \to \alpha \qquad \text{if } a \neq \{*\} \text{ or } \alpha \neq *$$
$$* = \{*\} \to *$$

P_∞ is not fully abstract for $\mathcal{H}^*$

[Park 1976]

Behaviours of D_∞ and P_∞

Hyperimmunity of D_∞

$$(a,\alpha) = a \to \alpha$$
$$\Cup$$
$$(b,\beta) = b \to \beta$$
$$\ddots$$
$$* = \emptyset \to *$$

Non-hyperimmunity of P_∞

$$* = \{*\} \to *$$
$$\Cup$$
$$* = \{*\} \to *$$
$$\Cup$$
$$* = \{*\} \to *$$
$$\ddots$$

Remark

By bijectivity of $\to$, every $\alpha \in D$ is an arrow.

In the λ-calculus, every term is a function

Example ($\overline{\omega}$)

Ordinal completion $\overline{\omega}$:

$$E_0 = \mathbb{N} \qquad\qquad n = \{i \mid i < n\} \to n$$

Hyperimmunity of $\overline{\omega}$

$$n_0 = [0, n_0[\to \cdots[0, n_0[\cdots \to n_0$$

$$\Cup\ (n_1 < n_0)$$

$$n_1 = [0, n_1[\to \cdots[0, n_1[\cdots \to n_1$$

$$\Cup\ (n_2 < n_1)$$

$$\ddots$$

$$0 = \emptyset \cdots \emptyset \cdots \emptyset \to 0$$

Example (D_∞^*)

Coppo,Dezani&Zacchi's D_∞^* (or Norm) : :

$$N_0 = \{p, q\} \qquad q = \{p\} \to q \qquad p = \{q\} \to p$$

D_∞^* is not fully abstract for $\mathcal{H}^*$ (but is for $\mathcal{H}$)

[Coppo,Dezani&Zacchi 1987]

Non-hyperimmunity of D_∞^*

$$p = \{q\} \to p$$
$$\Uplus$$
$$q = \{p\} \to q$$
$$\Uplus$$
$$p = \{q\} \to p$$
$$\ddots$$

Table of contents

Characterization

Hyperimmune models

A K-model D is *hyperimmune* when, $\forall(\alpha_n)_{n\geq 0}, \forall g : \mathbb{N} \to \mathbb{N}$,

if for all n :

$$\alpha_n = a_{n,1}\to \cdots \to a_{n,g(n)}\to \alpha'_n \quad \text{and} \quad \alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}.$$

then g is not recursive

- Generalization of Manzonetto's stratification.
- Intrinsically logically complex property.
- Link with recursion theory (hyperimmune functions).

Some intuitions

$$\alpha_1 = a_{1,1} \to \cdots a_{1,i_1} \cdots \to a_{1,g(1)} \to \alpha'_1$$

$$\Cup$$

$$\alpha_2 = a_{2,1} \to \cdots a_{2,i_2} \cdots \to a_{2,g(2)} \to \alpha'_2$$

$$\Cup$$

$$\alpha_3 = a_{3,1} \to \cdots a_{3,i_3} \cdots \to a_{3,g(3)} \to \alpha'_3$$

$$\ddots$$

Is D hyperimmune ?

If g is recursive and $g(k) \geq i_k$ for all k, then D is not hyperimmune (even if $(i_n)_n$ may not be recursive).

Examples (well-stratified)

Reminder on well-stratification :

It is the approximation by projections $(|.|_k)_{k\in\mathbb{N}}$ s.t.

$$|f|_{k+1}(x_k) = |f(x)|_k \qquad |f|_0(\bot) = |f(\bot)|_0$$

(Equivalent presentation of) Well-stratified K-models :

They are extensional completions of (where σ is a permutations)

$$U_0^{A,\sigma} = A \qquad \forall\alpha\in A,\quad \alpha = \emptyset\to\sigma(\alpha)$$

The completion preserves hyperimmunity

A completion $\bar{E}$ is hyperimmune iff E is.

Well-stratified models are hyperimmune

for all $\alpha_1\in A$ and $g:\mathbb{N}\to\mathbb{N}$,

$$\alpha_1 = \emptyset\to\cdots\to\emptyset\to\sigma^n(\alpha_1)$$

False ordinal completion ($\overline{\mathbb{N}}$)

Reminder of the ordinal completion $\overline{\omega}$:

$$E_0 = \mathbb{N} \qquad n = [0, n[\rightarrow n$$

False ordinal completion $\overline{\mathbb{N}}$:

$$E_0 = \mathbb{N} \qquad n = [0, n[\rightarrow (n+1)$$

Norm is not hyperimmune

$$g = (n \mapsto 2)$$

$$(\alpha_n)_n = (n, n, n, ...)$$

$$\begin{aligned} n &= [0, n[\rightarrow (n+1) \\ &= [0, n[\rightarrow [0, n] \rightarrow (n+2) \end{aligned}$$

$$\Cup$$

$$\begin{aligned} n &= [0, n[\rightarrow (n+1) \\ &= [0, n[\rightarrow [0, n] \rightarrow (n+2) \end{aligned}$$

$$\ddots$$

Example (H^f)

The non-well founded H^f :

$$H_0^f = \{*, \alpha_n \mid n\} \quad * = \emptyset \to * \quad \alpha_n = \underbrace{\emptyset \to \cdots \to}_{f(n)} \emptyset \to \{\alpha_{n+1}\} \to *$$

H^f is hyperimmune iff there is g st :

$$\alpha_0 = \underbrace{\emptyset \to \cdots}_{f(0)} \emptyset \to \underset{\Cup}{\{\alpha_1\}} \to \underbrace{\emptyset \quad \cdots}_{g(0)-f(0)} \to *$$

$$\alpha_1 = \underbrace{\emptyset \to \cdots}_{f(1)} \emptyset \to \underset{\Cup}{\{\alpha_2\}} \to \underbrace{\emptyset \quad \cdots}_{g(1)-f(1)} \to *$$

$$\alpha_2 = \underbrace{\emptyset \to \cdots}_{f(2)} \emptyset \to \{\alpha_3\} \cdots$$

$$\ddots$$

H^f is hyperimmune iff f is not below any recursive function g.

Our main result

Theorem

For any extensional K-models D, the following are equivalent when D respects approximation theorem :

- D is hyperimmune,
- D is inequationally fully abstract for $\mathcal{H}^*$,
- D is fully abstract for $\mathcal{H}^*$.

The proof needs some control over the assertion $\alpha \in [\![M]\!]$

The λ-calculus with D-tests $\Lambda_{\tau(D)}$

$$\tau_\alpha(M)\Downarrow \Leftrightarrow \alpha \in [\![M]\!]$$

Tests : procedural of $\tau_\alpha(M)$

$$\Gamma \vdash M : \alpha$$

Tests : procedural of $\tau_\alpha(M)$

$$\frac{\Gamma \vdash \lambda x_1 ... x_n . x_k \ N_1 \cdots N_m : \alpha}{\Gamma \vdash M : \alpha} \ M \to_h^* \lambda x_1 ... x_n . x_k \ N_1 \cdots N_m$$

Tests : procedural of $\tau_\alpha(M)$

$$\dfrac{\dfrac{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : \alpha}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m$$

Tests : procedural of $\tau_\alpha(M)$

$$
\dfrac{\dfrac{\dfrac{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : \alpha}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m
$$

Tests : procedural of $\tau_\alpha(M)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', x'_k : \beta \vdash x'_k\ N_1 \cdots N_m : \alpha'}{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}\ \exists\beta \in a_k}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : \alpha}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m
$$

Tests : procedural of $\tau_\alpha(M)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', x'_k : b_1 \to \cdots \to b_m \to \beta' \vdash x'_k\ N_1 \cdots N_m : \alpha'}{\Gamma', x'_k : \beta \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \beta = b_1 \to \cdots \to b_m \to \beta'}{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}\ \exists \beta \in a_k}{\dfrac{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m : \alpha}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 ... x_n . x_k\ N_1 \cdots N_m
$$

Tests : procedural of $\tau_\alpha(M)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma' \vdash N_i : \gamma_i \qquad \alpha' \leq \beta'}{\Gamma', x'_k : b_1 \to \cdots \to b_m \to \beta' \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \forall i, \forall \gamma_i \in b_i}{\Gamma', x'_k : \beta \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \beta = b_1 \to \cdots \to b_m \to \beta'}{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}\ \exists \beta \in a_k}{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\dfrac{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : \alpha}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'$$

Tests : procedural of $\tau_\alpha(M)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\Gamma' \vdash N_i : \gamma_i} \qquad \alpha' \leq \beta'}{\Gamma', x'_k : b_1 \to \cdots \to b_m \to \beta' \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \forall i, \forall \gamma_i \in b_i}{\Gamma', x'_k : \beta \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \beta = b_1 \to \cdots \to b_m \to \beta'}{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}\ \exists \beta \in a_k}{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\dfrac{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : \alpha}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'
$$

Tests : procedural of $\tau_\alpha(M)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\Gamma' \vdash N_i : \gamma_i} \quad \alpha' \leq \beta'}{\Gamma', x'_k : b_1 \to \cdots \to b_m \to \beta' \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \forall i, \forall \gamma_i \in b_i}{\Gamma', x'_k : \beta \vdash x'_k\ N_1 \cdots N_m : \alpha'}\ \beta = b_1 \to \cdots \to b_m \to \beta'}{\Gamma' \vdash x_k\ N_1 \cdots N_m : \alpha'}\ \exists \beta \in a_k}{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : a_1 \to \cdots \to a_n \to \alpha'}\ \Gamma' = (\Gamma, (x_i : a_i)_i)}{\Gamma \vdash \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m : \alpha}\ \alpha = a_1 \to \cdots \to a_n \to \alpha'}{\Gamma \vdash M : \alpha}\ M \to_h^* \lambda x_1 \ldots x_n . x_k\ N_1 \cdots N_m$$

4 possible failures

- M diverges
- $a_k = \emptyset$
- $\alpha' \not\leq \beta'$
- infinite derivation

Consistance

This procedure succeeds iff $\alpha \in [\![M]\!]^\Gamma$

by the approximation theorem's hypothesis

$\Lambda_{\tau(D)}$

The λ-calculus with D-tests internalises this reduction :

$$\tau_\alpha(M)\Downarrow \Leftrightarrow \alpha \in [\![M]\!]$$

Conclusion and questions

Theorem

For all extensional K-models D, the following are equivalent when it respects approximation theorem :

- D is hyperimmune,
- D is inequationally fully abstract for $\mathcal{H}^*$,
- D is fully abstract for $\mathcal{H}^*$.

- Is that a consequence of the expressiveness of K-models ?
  $\rightsquigarrow$ There is $2^{\aleph_0}$ λ-theories between $\mathcal{H}$ and $\mathcal{H}^*$...
- A link with the range property for $\mathcal{H}^*$?
  $\rightsquigarrow$ The uses of recursivity is reminiscent of [Polonsky 2012]

Definition of the category $\text{SCOTTL}_!$

A model of linear logic : SCOTTL

Objects : Posets **Morphisms :** linear fct. $\mathcal{I}(D) \to \mathcal{I}(P)$

Exponential : Finite antichains $!D = \mathcal{A}_f(D)$

$\mathcal{I}(D)$ represents the complete lattice of initial segments over D
a function is said linear if it preserves every sups

The Kleisli category : $\text{SCOTTL}_!$

Objects : Posets **Morphisms :** continuous fct. $\mathcal{I}(D) \to \mathcal{I}(P)$

Identities : $1_D = id_{\mathcal{I}(D)}$

Composition : the function composition

Cartesian product : $\&_{i\in I} D_i := \{(i,\alpha) | i \in I, \alpha \in A_D\}$

Exponential object : $A \Rightarrow B = \mathcal{A}_f(A)^{op} \times B$