ON THE CHARACTERIZATION OF MODELS OF $\mathcal{H}^*$: THE SEMANTICAL ASPECT

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Abstract. We give a characterization, with respect to a large class of models of untyped λ-calculus, of those models that are fully abstract for head-normalization, *i.e.*, whose equational theory is $\mathcal{H}^*$ (observations for head normalization). An extensional K-model D is fully abstract if and only if it is hyperimmune, *i.e.*, not well founded chains of elements of D cannot be captured by any recursive function.

This article, together with its companion paper [8] form the long version of [10]. It is a standalone paper that presents a purely semantical proof of the result as opposed to its companion paper that presents an independent and purely syntactical proof of the same result.

Introduction

The histories of full abstraction and denotational semantics of λ-calculi are both rooted in four fundamental articles published in the course of one year.

In 1976, Hyland [22] and Wadsworth [35] independently[1] proved the first full abstraction result of Scott's D_∞ (reflexive Scott's domain) for $\mathcal{H}^*$ (observations for head normalization). The following year, Milner [27] and Plotkin [32] showed respectively that PCF (a Turing-complete extension of the simply typed λ-calculus) has a unique fully abstract model up to isomorphism and that this model is not in the category of Scott domains and continuous functions.

Later, various articles focused on circumventing Plotkin counter-example [1, 21] or investigating full abstraction results for other calculi [2, 25, 30]. However, hardly anyone pointed out the fact that Milner's uniqueness theorem is specific to PCF, while $\mathcal{H}^*$ has various models that are fully abstract but not isomorphic.

The quest for a general characterization of the fully abstract models of head normalization started by successive refinements of a sufficient, but not necessary condition [14, 19, 26], improving the proof techniques from 1976 [22, 35]. While these results shed some light on various fully abstract semantics for $\mathcal{H}^*$, none of them could reach a full characterization.

In this article, we give the first full characterization of the full abstraction of an observational semantics for a specific (but large) class of models. The class we choose is that of Krivine-models, or K-models [24, 6]. This class, described in Section 1.2, is essentially the subclass of Scott complete lattices (or filter models [12]) which are prime algebraic. We add two further conditions:

Partially founded by French ANR project COQUAS (number 12 JS02 006 01).

[1] The idea already appears in Wadsworth thesis 3 years earlier.

DOI:10.2168/LMCS-???

extensionality and approximability of Definition 2.27. Extensionality is a standard and perfectly understood notion that requires the model to respect the η-equivalence. Notice that it is a necessary condition for the full abstraction if $\mathcal{H}^*$. Approximability is another standard notion saying that the model reflects the fact that a term is approximated by its finite Böhm trees. This notion has been extensively studied [3, Section III.17.3].

The extensional and approximable K-models are the objects of our characterization and can be seen as a natural class of models obtained from models of linear logic [18]. Indeed, the extensional K-models correspond to the extensional reflexive objects of the co-Kleisli category associated with the exponential comonad of Ehrhard's ScottL category [15] (Prop. 1.10).

We achieve the characterization of full abstraction for $\mathcal{H}^*$ in Theorem 1.23: a model D is fully abstract for $\mathcal{H}^*$ iff D is *hyperimmune* (Def. 1.19). Hyperimmunity is the key property our study introduces in denotational semantics. This property is reminiscent of the Post's notion of hyperimmune sets in recursion theory. Hyperimmunity in recursion theory is not only undecidable, but also surprisingly high in the hierarchy of undecidable properties (it cannot be decided by a machine with an oracle deciding the halting problem) [29].

Roughly speaking, a model D is hyperimmune whenever the λ-terms can have access to only well-founded[2] chains of elements of D. In other words, D might have non-well-founded chains $d_0 \geq d_1 \geq \cdots$, but these chains "grow" so fast (for a suitable notion of growth), that they cannot be contained in the interpretation of any λ-term.

The intuition that full abstraction of $\mathcal{H}^*$ is related to a kind of well-foundedness can be found in the literature (*e.g.*, Hyland's [22], Gouy's [19] or Manzonetto's [26]). Our contribution is to give, with hyperimmunity, a precise definition of this intuition, at least in the setting of K-models.

A finer intuition can be described in terms of game semantics. Informally, a game semantics for the untyped λ-calculus takes place in the arena interpreting the recursive type $o = o \to o$. This arena is infinitely wide (by developing the left o) and infinitely deep (by developing the right o). Moves therein can thus be characterized by their nature (question or answer) and by a word over natural numbers. For example, $q(2.3.1)$ represents a question in the underlined "o" in $o = o{\to}(o{\to}o{\to}(\underline{o}{\to}o){\to}o){\to}o$. Plays in this game are potentially infinite sequences of moves, where a question of the form $q(w)$ is followed by any number of deeper questions/answers, before an answer $a(w)$ is eventually provided, if any.

A play like $q(\epsilon), q(1)...a(1), q(2)...a(2), q(3)...$ is admissible: one player keeps asking questions and is infinitely delaying the answer to the initial question, but some answers are given so that the stream is productive. However, the full abstraction for $\mathcal{H}^*$ forbids non-productive infinite questioning like in $q(\epsilon), q(1), q(1.1), q(1.1.1)...$, in general. Nevertheless, disallowing *all* such strategies is sufficient, but not necessary to get full abstraction. The hyperimmunity condition is finer: non productive infinite questioning is allowed *as long as* the function that chooses the next question grows faster than any recursive function (notice that in the example above that choice is performed by the constant $(n \mapsto 1)$ function). For example, if $(u_i)_{i\geq 0}$ grows faster than any recursive function, the play $q(\epsilon), q(u_1), q(u_1.u_2), q(u_1.u_2.u_3)...$ is perfectly allowed.

Incidentally, we obtain a significant corollary (also expressed in Theorem 1.23) stating that full abstraction coincides with inequational full abstraction for $\mathcal{H}^*$ (equivalence between observational and denotational orders). This is in contrast to what happens to other calculi [34, 16].

In the literature, most of the proofs of full abstraction for $\mathcal{H}^*$ are based on Nakajima trees [28] or some other notion of quotient of the space of Böhm trees, using the characterization of the observational equivalence (see Proposition 2.13). The usual approach is too coarse because it

[2] well-foundedness is considered with regard to a new order independent from the poset order of D.

considers arbitrary Böhm trees which are not necessarily images of actual λ-terms. To overcome this we propose two different techniques leading to two different proofs of the main result: one purely semantical and the other purely syntactical. In this article we only present the former, the latter being the object of a companion paper [8].

This proof follows the line of historical ones while overcoming weaknesses of Nakajima trees with a notion of quasi-approximation property (Def. 2.32), that involves recursivity in a refined way. Quasi-approximability is a key tool in the proof, which is otherwise quite standard. However, since Böhm trees are specific to the λ-calculus and head reduction, there is not much hope to extend the proof to many other calculi/strategies (such as differential λ-calculus [17], or call-by-value strategies).

1. Preliminaries and result

1.1. Preliminaries.

1.1.1. *Preorders.*

Given two partially ordered sets $D = (|D|, \leq_D)$ and $E = (|E|, \leq_E)$, we denote:

- $D^{op} = (|D|, \geq_D)$ the reverse-ordered set.
- $D \times E = (|D| \times |E|, \leq_{D\times E})$ the Cartesian product endowed with the pointwise order:
$$(\delta, \epsilon) \leq_{D\times E} (\delta', \epsilon') \quad \text{if} \quad \delta \leq_D \delta' \quad \text{and} \quad \epsilon \leq_E \epsilon'.$$
- $\mathcal{A}_f(D) = (|\mathcal{A}_f(D)|, \leq_{\mathcal{A}_f(D)})$ the set of finite antichains of D (*i.e.*, finite subsets whose elements are pairwise incomparable) endowed with the order :
$$a \leq_{\mathcal{A}_f(D)} b \;\Leftrightarrow\; \forall \alpha \in a, \exists \beta \in b, \alpha \leq_D \beta$$

In the following will we use D for $|D|$ when there is no ambiguity. Initial Greek letters α, β, γ... will vary on elements of ordered sets. Capital initial Latin letters A, B, C... will vary over subsets of ordered sets. And finally, initial Latin letters a, b, c... will denote finite antichains.

An *order isomorphism* between D and E is a bijection $\phi : |D| \to |E|$ such that ϕ and ϕ^{-1} are monotone.

Given a subset $A \subseteq |D|$, we denote $\downarrow A = \{\alpha \mid \exists \beta \in A, \alpha \leq \beta\}$. We denote by $\mathcal{I}(D)$ the set of *initial segments of* D, that is $\mathcal{I}(D) = \{\downarrow A \mid A \subseteq |D|\}$. The set $\mathcal{I}(D)$ is a prime algebraic complete lattice with respect to the set-theoretical inclusion. The *sups* are given by the unions and the *prime elements* are the downward closure of the singletons. The *compact elements* are the downward closure of finite antichains.

The domain of a partial function f is denoted by $Dom(f)$. The *graph* of a Scott-continuous function $f : \mathcal{I}(D) \to \mathcal{I}(E)$ is

$$\text{graph}(f) = \{(a, \alpha) \in \mathcal{A}_f(D)^{op} \times E \mid \alpha \in f(\downarrow a)\} \tag{1.1}$$

Notice that elements of $\mathcal{I}(\mathcal{A}_f(D)^{op} \times E)$ are in one-to-one correspondence with the graphs of Scott-continuous functions from $\mathcal{I}(D)$ to $\mathcal{I}(E)$.

1.1.2. *λ-calculus.*
The λ-terms are defined up to α-equivalence by the following grammar using notation "*à la* Barendregt" [4] (where variables are denoted by final Latin letters x, y, z...):

$$\Lambda \quad (\lambda\text{-terms}) \quad M, N \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad M\,N$$

We denote by $\mathrm{FV}(M)$ the set of free variables of a λ-term M. Moreover, we abbreviate a nested abstraction $\lambda x_1...x_k.M$ to $\lambda \vec{x}^k M$, or, when k is irrelevant, to $\lambda \vec{x} M$. We denote by $M[N/x]$ the capture-free substitution of x by N.
The λ-terms are subject to the β-reduction:

$$(\beta) \qquad (\lambda x.M)\,N \xrightarrow{\beta} M[N/x]$$

A context C is a λ-term with possibly some occurrences of a hole, *i.e.*:

$$\Lambda^{(\!|.|\!)} \quad (\text{contexts}) \quad C \quad ::= \quad (\!|.|\!) \quad | \quad x \quad | \quad \lambda x.C \quad | \quad C_1\,C_2$$

The writing $C(\!|M|\!)$ denotes the term obtained by filling the holes of C by M. The small step reduction $\to$ is the closure of (β) by any context, and $\to_h$ is the closure of (β) by the rules:

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'} \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\,N \to_h M'\,N}$$

The transitive reduction $\to^*$ (resp $\to_h^*$) is the reflexive transitive closure of $\to$ (resp $\to_h$).
The big step head reduction, denoted $M{\Downarrow^{\mathrm{h}}}N$, is $M \to_h^* N$ for N in a *head-normal form*, *i.e.*, $N = \lambda x_1...x_m.y\ M_1 \cdots M_n$, for $M_1, ..., M_m$ any terms. We write $M{\Downarrow^{\mathrm{h}}}$ for the (*head*) *convergence*, *i.e.*, whenever there is N such that $M{\Downarrow^{\mathrm{h}}}N$.

Example 1.1.
- The *identity term* $\boldsymbol{I} := \lambda x.x$ takes a term and returns it as it is:
$$\boldsymbol{I}\,M \quad \to \quad M.$$
- The n^{th} Church numeral, denoted by $\underline{n}$, and the successor function, denoted by $\boldsymbol{S}$, are defined by
$$\underline{n} := \lambda f x.\underbrace{f\,(f \cdots\ f\,(f}_{n \text{ times}}\ x) \cdots), \qquad\qquad \boldsymbol{S} := \lambda u f x.u\ f\ (f\ x).$$
Together they provide a suitable encoding for natural numbers, with $\underline{n}$ representing the n^{th} iteration.
- The *looping term* $\boldsymbol{\Omega} := (\lambda x.xx)\ (\lambda x.xx)$ infinitely reduces into itself, notice that $\boldsymbol{\Omega}$ is an example of a diverging term:
$$\boldsymbol{\Omega} \quad \to \quad (x\,x)[\lambda y.y\,y/x] \quad = \quad \boldsymbol{\Omega} \quad \to \quad \boldsymbol{\Omega} \quad \to \quad \cdots.$$
- The *Turing fixpoint combinator* $\boldsymbol{\Theta} := (\lambda uv.v\ (u\ u\ v))\ (\lambda uv.v\ (u\ u\ v))$ is a term that computes the least fixpoint of its argument (if it exists):
$$\begin{aligned}\boldsymbol{\Theta}\ M &\to (\lambda v.v\ ((\lambda uv.v\ (u\ u\ v))\ (\lambda uv.v\ (u\ u\ v))v))\ M\\ &= (\lambda v.v\ (\boldsymbol{\Theta}\ v))\ M\\ &\to M\ (\boldsymbol{\Theta}\ M).\end{aligned}$$

Other notions of convergence exist (strong, lazy, call by value...), but our study focuses on head convergence, inducing the equational theory denoted by $\mathcal{H}^*$.

Definition 1.2. The *observational preorder* and *equivalence* denoted $\sqsubseteq_{\mathcal{H}^*}$ and $\equiv_{\mathcal{H}^*}$ are given by:

$$\begin{array}{lll} M \sqsubseteq_{\mathcal{H}^*} N & \text{if} & \forall C,\ C(\!|M|\!){\Downarrow^{\mathrm{h}}} \ \Rightarrow\ C(\!|N|\!){\Downarrow^{\mathrm{h}}},\\ M \equiv_{\mathcal{H}^*} N & \text{if} & M \sqsubseteq_{\mathcal{H}^*} N \text{ and } N \sqsubseteq_{\mathcal{H}^*} M.\end{array}$$

The resulting (in)equational theory is called $\mathcal{H}^*$.

Definition 1.3. A model of the untyped λ-calculus with an interpretation $[\![-]\!]$ is:

- fully abstract (for $\mathcal{H}^*$) if for all $M, N \in \Lambda$:
$$M \equiv_{\mathcal{H}^*} N \quad \text{if} \quad [\![M]\!] = [\![N]\!],$$
- inequationally fully abstract (for $\mathcal{H}^*$) if for all $M, N \in \Lambda$:[3]
$$M \sqsubseteq_{\mathcal{H}^*} N \quad \text{if} \quad [\![M]\!] \subseteq [\![N]\!].$$

Henceforth, convergence of a λ-term means head convergence, and full abstraction for λ-calculus means full abstraction for $\mathcal{H}^*$.

Concerning recursive properties of λ-calculus, we will use the following one:

Proposition 1.4 ([4, Proposition 8.2.2] [4])**.**
Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of terms such that:

- $\forall n \in \mathbb{N}, M_n \in \Lambda^0$,
- *the encoding of $(n \mapsto M_n)$ is recursive,*

then there exists F such that:
$$\forall n, F\ \underline{n} \ \to^* M_n.$$

1.2. **K-models.**

We introduce here the main semantical object of this article: extensional K-models [24][6]. This class of models of the untyped λ-calculus is a subclass of filter models [12] containing many extensional models from the continuous semantics, like Scott's D_∞ [33].

1.2.1. *The category* ScottL$_!$.

Extensional K-models correspond to the extensional reflexive Scott domains that are prime algebraic complete lattices and whose application embeds prime elements into prime elements [20, 36]. However we prefer to exhibit K-models as the extensional reflexive objects of the category ScottL$_!$ which is itself the Kleisli category over the linear category ScottL [15].

Definition 1.5. We define the Cartesian closed category ScottL$_!$ [20, 36, 15]:

- *objects* are partially ordered sets.
- *morphisms* from D to E are Scott-continuous functions between the complete lattices $\mathcal{I}(D)$ and $\mathcal{I}(E)$.

The *Cartesian product* is the disjoint sum of posets. The *terminal object* $\top$ is the empty poset. The *exponential object* $D{\Rightarrow}E$ is $\mathcal{A}_f(D)^{op}{\times}E$. Notice that an element of $\mathcal{I}(D{\Rightarrow}E)$ is the graph of a morphism from D to E (see Equation (1.1)). This construction provides a natural isomorphism between $\mathcal{I}(D{\Rightarrow}E)$ and the corresponding homset. Notice that if $\simeq$ denotes isomorphisms in ScottL$_!$, then:
$$D \Rightarrow D \Rightarrow \cdots \Rightarrow D \simeq (\mathcal{A}_f(D)^{op})^n \times D. \tag{1.2}$$
For example $D \Rightarrow (D \Rightarrow D) \simeq \mathcal{A}_f(D)^{op} \times (\mathcal{A}_f(D)^{op} \times D) = (\mathcal{A}_f(D)^{op})^2 \times D$.

[3] This can be generalised by replacing $\subseteq$ by any order on the model.

[4] This is not the original statement. We remove the dependence on $\vec{x}$ that is empty in our case and we replace the β-equivalence by a reduction since the proof of Barendregt [4] works as well with this refinement.

Remark 1.6. In the literature (*e.g.* [20, 36, 15]), objects are preodered sets and the exponential object $D \Rightarrow D$ is defined by using finite subsets (or multisets) instead of the finite antichains. Our presentation is the quotient of the usual one by the equivalence relation induced by the preorder. The two presentations are equivalent (in terms of equivalence of category) but our choice simplifies the definition of hyperimmunity (Definition 1.19).

Proposition 1.7. *The category* ScottL$_!$ *is isomorphic to the category of prime algebraic complete lattices and Scott-continuous maps.*

Proof. Given a poset D, the initial segments $\mathcal{I}(D)$ form a prime algebraic complete lattice which prime elements are the downward closures $\downarrow\alpha$ of any $\alpha \in D$ since $I = \bigcup_{\alpha\in I}\downarrow\alpha$. Conversely, the prime elements of a prime algebraic complete lattice form a poset. The two operations are inverse one to the other modulo ScottL$_!$-isomorphisms and Scott-continuous isomorphisms. □

1.2.2. *An algebraic presentation of K-models.*

Definition 1.8 ([24])**.** An *extensional K-model* is a pair (D, i_D) where:

- D is a poset.
- i_D is an order isomorphism between $D{\Rightarrow}D$ and D.

By abuse of notation we may denote the pair (D, i_D) simply by D when it is clear from the context we are referring to an extensional K-model.

Definition 1.9. Given a Cartesian closed category C, an extensional reflexive objects of C is an objects D endowed with an isomorphism $abs_D : (D \Rightarrow D) \to D$ (and $app_D := abs_D^{-1}$). This corresponds to the categorical axiomatisation of extensional models of the untyped λ-calculus.

Proposition 1.10. *Extensional K-models correspond exactly to extensional reflexive objects of* ScottL$_!$.

Proof. Given a K-model (D, i_D), the isomorphism between $D{\Rightarrow}D$ and D is given by:

$$\forall A \in \mathcal{I}(D{\Rightarrow}D), \qquad \mathrm{app}_D(A) = \{i_D(a,\alpha) \mid (a,\alpha) \in A\},$$
$$\forall B \in \mathcal{I}(D), \qquad \mathrm{abs}_D(B) = \{(a,\alpha) \mid i_D(a,\alpha) \in B\}.$$

Conversely, consider an extensional reflexive object (D, app_D, abs_D) of ScottL$_!$. Since abs_D is an isomorphism, it is linear (that is, it preserves all sups). For all $(a,\alpha) \in D{\Rightarrow}D$, we have

$$\downarrow(a,\alpha) = abs(app(\downarrow(a,\alpha))) = \bigcup_{\beta\in app(\downarrow(a,\alpha))} abs(\downarrow\beta).$$

Thus there is $\beta \in app(\downarrow(a,\alpha)$ such that $(a,\alpha) \in abs(\downarrow\beta)$, and since $abs(\downarrow\beta) \subseteq \downarrow(a,\alpha)$, this is an equality. Thus there is a unique β such that $app_D(a,\alpha) = \downarrow\beta$, this is $i_D(a,\alpha)$. □

In the following we will not distinguish between a K-model and its associated reflexive object, this is a model of the pure λ-calculus.

Definition 1.11. An *extensional partial K-model* is a pair (E, j_E) where E is an object of ScottL$_!$ and j_E is a partial function from $E{\Rightarrow}E$ to E that is an order isomorphism between $\mathrm{Dom}(j_E)$ and E.

$$E \quad \overset{j_E}{\longleftrightarrow} \quad \mathrm{Dom}(j_E) \quad \subseteq \quad (E \Rightarrow E)$$

Definition 1.12. The *completion of a partial K-model* (E, j_E) is the union

$$(\bar{E}, j_{\bar{E}}) = (\bigcup_{n\in\mathbb{N}} E_n, \bigcup_{n\in\mathbb{N}} j_{E_n})$$

of partial completions (E_n, j_{E_n}) that are extensional partial K-models defined by induction on n. We define $(E_0, j_{E_0}) := (E, j_E)$ and:

- $|E_{n+1}| \ := \ |E_n| \cup (|E_n \Rightarrow E_n| - Dom(j_{E_n}))$
- $j_{E_{n+1}}$ is defined only over $|E_n \Rightarrow E_n| \subseteq |E_{n+1} \Rightarrow E_{n+1}|$ by $j_{E_{n+1}} := j_{E_n} \cup id_{|E_n \Rightarrow E_n| - Dom(j_{E_n})}$
- $\leq_{E_{n+1}}$ is given by $j_{E_{n+1}}(a,\alpha) \leq_{E_{n+1}} (b,\beta)$ if $a \geq_{\mathcal{A}_f(E_n)} b$ and $\alpha \leq_{E_n} \beta$.

Remark that E_{n+1} corresponds to $E_n \Rightarrow E_n$ up to isomorphism, what leads to the equivalent definition:

Proposition 1.13. *The completion $(\bar{E}, j_{\bar{E}})$ of an extensional partial K-model (E, j_E) can be described as the categorical ω-colimit (in* ScottL*) of $(E'_n)_n$ along the injections $(j_n^{-1})_n$. The posets $(E'_n)_n$ and the partial functions $(j_n)_n$ are defined by induction by $(E'_0, j_0) := (E, j_E)$, and for $n \geq 0$, by $E'_{n+1} := E'_n \Rightarrow E'_n$ and for all $a \subseteq dom(j_n)$ and $\alpha \in j_n$, $j_{n+1}(a,\alpha) := (j_n(a), j_n(\alpha))$.*

$$\begin{tikzcd}
& & \overline{E} & & & \\
E \arrow[r, "j_E^{-1}"'] \arrow[urr] & E_1 \arrow[r, "j_1^{-1}"'] \arrow[ur] & E_2 \arrow[r, "j_2^{-1}"'] \arrow[u] & \cdots \arrow[r, "j_{n-1}^{-1}"'] & E_n \arrow[r, "j_n^{-1}"'] \arrow[ull] & \cdots
\end{tikzcd}$$

Remark 1.14. The completion of an extensional partial K-model (E, j_E) is the smallest extensional K-model $\bar{E}$ containing E. In particular, any extensional K-model D is the extensional completion of itself: $D = \bar{D}$.

Example 1.15.

(1) *Scott's D_∞* [33] is the extensional completion of

$$|D| := \{*\}, \qquad \leq_D := id, \qquad j_D := \{(\emptyset, *) \mapsto *\}.$$

The completion the a triple $(|D_\infty|, \leq_{D_\infty}, j_{D_\infty})$ where $|D_\infty|$ is generated by:

$$\begin{array}{llll} |D_\infty| & \alpha,\beta & ::= & * \ \mid \ a{\rightarrow}\alpha \\ |!D_\infty| & a, b & \in & \mathcal{A}_f(|D_\infty|) \end{array}$$

except that $\emptyset{\rightarrow}* \notin |D_\infty|$; j_{D_∞} is defined by $j_{D_\infty}(\emptyset, *) = *$ and $j_{D_\infty}(a,\alpha) = a{\rightarrow}\alpha$ for $(a,\alpha) \neq (\emptyset, *)$.

(2) *Park's P_∞* [31] is the extensional completion of

$$|P| := \{*\}, \qquad \leq_P := id, \qquad j_P := \{(\{*\}, *) \mapsto *\};$$

i.e., $|P_\infty|$ is defined by the previous grammar except that $(\{*\}{\rightarrow}*) \notin |P_\infty|$ while $\emptyset{\rightarrow}* \in |P_\infty|$.

(3) *Norm* or D_∞^* [13] is the extensional completion of

$$|E| := \{p, q\}, \qquad \leq_E := id \cup \{p < q\},$$
$$j_E := \{(\{p\}, q){\mapsto}q, (\{q\}, p){\mapsto}p\}.$$

(4) *Well-stratified K-models* [26] are the extensional completions of some E respecting

$$\forall (a,\alpha){\in}\mathrm{Dom}(j_E), a{=}\emptyset.$$

(5) The *inductive* $\overline{\omega}$ is the extensional completion of

$$|E| := \mathbb{N}, \qquad \leq_E := id, \qquad j_E := \{(\{k \mid k < n\}, n){\mapsto}n \mid n \in \mathbb{N}\}.$$

$$[\![x_i]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \alpha \leq \beta \in a_i\} \qquad [\![\lambda y.M]\!]_D^{\vec{x}} = \{(\vec{a}, b{\to}\alpha) \mid (\vec{a}b,\alpha) \in [\![M]\!]_D^{\vec{x}y}\}$$
$$[\![M\,N]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \exists b, (\vec{a}, b{\to}\alpha) \in [\![M]\!]_D^{\vec{x}} \ \wedge \forall \beta{\in}b, (\vec{a},\beta) \in [\![N]\!]_D^{\vec{x}}\}$$

FIGURE 1. Direct interpretation of Λ in D

(6) The *co-inductive* $\overline{\mathbb{Z}}$ is the extensional completion of

$$|E| := \mathbb{Z}, \qquad \leq_E := id, \qquad j_E := \{(\{n\},\ n{+}1){\mapsto}n{+}1 \mid n \in \mathbb{Z}\}.$$

(7) *Functionals H^f* (given $f : \mathbb{N} \to \mathbb{N}$) are the extensional completions of:

$$|E| := \{*\} \cup \{\alpha_j^n \mid n \geq 0,\ 1 \leq j \leq f(n)\}, \qquad \leq_E := id,$$
$$j_E := \Big\{(\emptyset, *) \mapsto *\Big\} \cup \Big\{(\emptyset, \alpha_{j+1}^n) \mapsto \alpha_j^n \mid 1 \leq j < f(n)\Big\} \cup \Big\{(\{\alpha_1^{n+1}\}, *) \mapsto \alpha_{f(n)}^n \mid n \in \mathbb{N}^*\Big\},$$

where $(\alpha_j^n)_{n,j}$ is a family of atoms different from $*$.

For the sake of simplicity, from now on we will work with a fixed extensional K-model D. Moreover, we will use the notation $a{\to}\alpha := i_D(a,\alpha)$. Notice that, due to the injectivity of i_D, any $\alpha \in D$ can be uniquely rewritten into $a{\to}\alpha'$, and more generally into $a_1{\to}\cdots{\to}a_n{\to}\alpha_n$ for any n.

Remark 1.16. Using these notations, the model H^f can be summarized by writing, for each n:

$$\alpha_1^n \ = \ \underbrace{\emptyset{\to}\cdots{\to}\emptyset}_{f(n)}{\to}\{\alpha_1^{n+1}\}{\to}*$$

1.2.3. *Interpretation of the λ-calculus.*

The Cartesian closed structure of ScottL$_!$ endowed with the isomorphisms app_D and abs_D of the reflexive object induced by D (see Proposition 1.10) defines, in a standard manner, a model of the λ-calculus.

A term M with at most n free variables $x_1,\ldots,x_n$ is interpreted as the graph of a morphism $[\![M]\!]_D^{x_1\ldots x_n}$ from D^n to D (when n is obvious, we can use $[\![.]\!]^{\bar{x}}$). By Equations (1.1) and (1.2) we have:

$$[\![M]\!]_D^{x_1\ldots x_n} \subseteq \ (D \Rightarrow \cdots \Rightarrow D \Rightarrow D) \ \simeq \ (\mathcal{A}_f(D)^{op})^n \times D.$$

In Figure 1, we explicit the interpretation $[\![M]\!]_D^{x_1\ldots x_n}$ by structural induction on M.

Example 1.17.

$$[\![\lambda x.y]\!]_D^y = \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in a\},$$
$$[\![\lambda x.x]\!]_D^y = \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in b\},$$
$$[\![\boldsymbol{I}]\!]_D = \{a{\to}\alpha \mid \alpha \leq_D \beta \in a\},$$
$$[\![\underline{1}]\!]_D = \{a{\to}b{\to}\alpha \mid \exists c, c{\to}\alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b\}.$$

In the last two cases, terms are interpreted in an empty environment. We omit the empty sequence associated with the empty environment, *e.g.*, $a{\to}b{\to}\alpha$ stands for $((), a{\to}b{\to}\alpha)$.
We can verify that extensionality holds, indeed $[\![\underline{1}]\!]_D = [\![\boldsymbol{I}]\!]_D$, since $c{\to}\alpha \leq_D \beta \in a$ and $c \leq_{\mathcal{A}_f(D)} b$ exactly say that $b{\to}\alpha {\leq_D} \beta {\in} a$, and since any element of $\gamma {\in} D$ is equal to $d{\to}\delta$ for a suitable d and δ.

$$\frac{\alpha \in a}{x : a \vdash x : \alpha}\ (I\text{-}id) \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : a \vdash M : \alpha}\ (I\text{-}weak) \qquad \frac{\Gamma \vdash M : \beta \qquad \alpha \leq \beta}{\Gamma \vdash M : \alpha}\ (I\text{-} \leq)$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a \to \alpha}\ (I\text{-}\lambda) \qquad \frac{\Gamma \vdash M : a \to \alpha \qquad \forall \beta \in a,\ \Gamma \vdash N : \beta}{\Gamma \vdash M\,N : \alpha}\ (I\text{-}@)$$

FIGURE 2. Intersection type system computing the interpretation in D

1.2.4. *Intersection types.*

It is folklore that the interpretation of the λ-calculus into a given K-model D is characterized by a specific *intersection type system*. In fact any element $\alpha \in D$ can be seen as an intersection type

$$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta \qquad\qquad \text{given by} \quad \alpha = \{\alpha_1, \ldots, \alpha_n\} \to \beta.$$

In Figure 2, we give the intersection-type assignment corresponding to the K-model induced by D.

Proposition 1.18. *Let M be a term of Λ, the following statements are equivalent:*
- $(\vec{a}, \alpha) \in [\![M]\!]_D^{\vec{x}}$,
- *the type judgment $\vec{x} : \vec{a} \vdash M : \alpha$ is derivable by the rules of Figure 2.*

Proof. By structural induction on the grammar of Λ. □

1.3. **The result.**

We state our main result, claiming an equivalence between hyperimmunity (Def. 1.19) and full abstraction for $\mathcal{H}^*$.

Definition 1.19 (*Hyperimmunity*)**.** A (possibly partial) extensional K-model D is said to be *hyperimmune* if for every sequence $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$, there is no recursive function $g : \mathbb{N} \to \mathbb{N}$ satisfying:

$$\forall n \geq 0, \qquad \alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n \qquad \text{and} \qquad \alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}. \tag{1.3}$$

Notice, in the above definition, that each antichain $a_{n,i}$ always exists and it is uniquely determined by the isomorphism between D and $D \Rightarrow D$ that allows us to unfold any element α_i as an arrow (of any length).

The idea is the following. The sequence $(\alpha_n)_{n\geq 0}$ is morally describing a non well-founded chain of elements of D, through the isomorphism $D \simeq D \Rightarrow D$, allowing us to see any element α_i as an arrow (of any length):

$$\begin{array}{lll}
\alpha_0 = a_{0,1} \to \cdots\ a_{0,i_0} \cdots \to a_{0,g(0)} \to \alpha'_0 & & \\
\qquad\qquad\quad\ \ \uplus & & \\
\qquad\qquad\quad\ \ \alpha_1 = a_{1,1} \to \cdots\ \ a_{1,i_1} \cdots \to a_{1,g(1)} \to \alpha'_1 & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ \uplus & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ \alpha_2 = a_{2,1} \to \cdots\ a_{2,i_2} \cdots \to a_{2,g(2)} \to \alpha'_2 & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ddots & &
\end{array}$$

The growth rate $(i_n)_n$ of the chain $(\alpha_n)_n$ depends on how many arrows must be displayed in α_i in order to see α_{i+1} as an element of the antecedent of one of them. Now, hyperimmunity means that if any such non-well founded chain $(\alpha_n)_n$ exists, then its growth rate $(i_n)_n$ cannot be bounded by any recursive function g.

Remark 1.20. It would not be sufficient to simply consider the function $n \mapsto i_n$ such that $\alpha_{n+1} \in a_{n,i_n}$ rather than the bounding function g. Indeed, $n \mapsto i_n$ may not be recursive even while g is.

Proposition 1.21. *For any extensional partial K-model E (Def. 1.11), the completion $\overline{E}$ (Def. 1.12) is hyperimmune iff E is hyperimmune.*

Proof. The left-to-right implication is trivial.
The right-to-left one is obtained by contradiction:
Assume to have a $(\alpha_n)_{n\geq 0} \in \bar{E}^{\mathbb{N}}$ and a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that for all $n \geq 0$:

$$\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n \qquad \text{and} \qquad \alpha_{n+1} \in \bigcup_{i \leq g(n)} a_{n,i}$$

Recall that the sequence $(E_k)_{k\geq 0}$ of Definition 1.12 approximates the completion $\bar{E}$.
Then we have the following:

- There exists k such that $\alpha_0 \in E_k$, because $\alpha_0 \in \bar{E} = \bigcup_k E_k$.
- If $\alpha_n \in E_{j+1}$, then $\alpha_{n+1} \in E_j$, because there is $i \leq g(n)$ such that $\alpha_{n+1} \in a_{n,i} \subseteq E_j$.
- If $\alpha_n \in E_0 = E$, then $\alpha_{n+1} \in E$ by surjectivity of j_E.

Thus there is k such that $(\alpha_n)_{n\geq k} \in E^{\mathbb{N}}$, which would break hyperimmunity of E. □

Example 1.22.
- The well-stratified K-models of Example 1.15(4) (and in particular D_∞ of Item (1)) are trivially hyperimmune: already in the partial K-model, there are not even α_1, α_2 and n such that $\alpha_1 = a_1 \to \cdots \to a_n \to \alpha'_1$ and $\alpha_2 \in a_n$ (since $a_n = \emptyset$). The non-hyperimmunity of the partial K-model can be extended to the completion using Proposition 1.21.
- The model $\overline{\omega}$ (Ex. 1.15(5)) is hyperimmune. Indeed, any such $(\alpha_n)_n$ in the partial K-model would respect $\alpha_{n+1} <_{\mathbb{N}} \alpha_n$, hence $(\alpha_n)_n$ must be finite by well-foundedness of $\mathbb{N}$.
- The models P_∞, D^*_∞ and $\overline{\mathbb{Z}}$ (Examples 1.15(2), (3) and (6)) are not hyperimmune. Indeed for all of them $g = (n \mapsto 1)$ satisfies the condition of Equation (1.3), the respective non-well founded chains $(\alpha_i)_i$ being $(*, *, \dots)$, $(p, q, p, q, \dots)$, and $(0, -1, -2, \dots)$:

$$\begin{array}{ccc}
* = \{*\} \to * & p = \{q\} \to p & 0 = \{1\} \to 0 \\
\Cup & \Cup & \Cup \\
* = \{*\} \to * & q = \{p\} \to q & 1 = \{2\} \to 1 \\
\Cup & \Cup & \Cup \\
* = \{*\} \to * & p = \{q\} \to p & 2 = \{3\} \to 2 \\
\ddots & \ddots & \ddots
\end{array}$$

- More interestingly, the model H^f (Ex. 1.15(7)) is hyperimmune iff f is a *hyperimmune function* [29], *i.e.*, iff there is no recursive $g : \mathbb{N} \to \mathbb{N}$ such that $f \leq g$ (pointwise order);

otherwise the corresponding sequence is $(\alpha_1^i)_i$.

$$\alpha_1^0 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(0)\ \text{times}} \to \{\alpha_1^1\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\alpha_1^1 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(1)\ \text{times}} \to \{\alpha_1^2\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\alpha_1^2 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(2)\ \text{times}} \to \{\alpha_1^3\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\ddots$$

The following theorem constitutes the main result of the paper. It shows the equivalence between hyperimmunity and (inequational) full abstraction for $\mathcal{H}^*$ under a certain condition. This condition, namely the approximation property, is a standard property that will be defined in more details in Definition 2.27.

Theorem 1.23. *For any extensional and approximable K-model D (Def. 2.27), the following are equivalent:*

(1) *D is hyperimmune,*
(2) *D is inequationally fully abstract for $\mathcal{H}^*$,*
(3) *D is fully abstract for $\mathcal{H}^*$.*

Example 1.24. The model D_∞ (Ex.1.15(1)), the model $\overline{\omega}$ (Ex.1.15(5)) and the well-stratified K-models (Ex.1.15(4)) will be shown inequationally fully abstract, as well as the models H^f when f is hyperimmune. The models D_∞^*, $\overline{\overline{\mathbb{Z}}}$ (Ex.1.15(3) and Ex.1.15(6)) will not be, as well as the model H^f for f not hyperimmune.

2. Proof

The main idea of this proof is not new, it consists in using Böhm trees to decompose the interpretation of the λ-calculus. In order to do so, we need to interpret them into our K-model D so that the following diagram commutes:

$$\begin{tikzcd} \Lambda \arrow[rr, "[\![.]\!]"] \arrow[dr, "\boldsymbol{BT}(.)"'] & & D \\ & \boldsymbol{BT} \arrow[ur, "[\![.]\!]_*"'] & \end{tikzcd}$$

The approximation and quasi-approximation properties of Definitions 2.27 and 2.32 exactly state this decomposition for two specific choices of interpretation. Indeed, we will see in Definition 2.22 that there are many different possible interpretations of the Böhm trees, we will mainly focus on the inductive interpretation (Def. 2.25) and the quasi-finite interpretation (Def. 2.31).

The approximation and quasi-approximation properties will have different roles. The approximation property, *i.e.*, the decomposition via the inductive interpretation, mainly says that the interpretation of terms is approximable by finite Böhm trees. Approximation property is a hypothesis of Theorem 1.23 and it holds in all known candidates to full abstraction, *i.e.*, extensional and sensible models (Ex. 2.29). We even conjecture, in fact, that all K-models that are fully abstract for $\mathcal{H}^*$ respect the approximation property.

The quasi-approximation property is a fairly finer property[5] that is based on deep references to recursivity theory. The quasi-approximation property will be proved equivalent to both full abstraction for $\mathcal{H}^*$ and hyperimmunity in the presence of the approximation property.

Theorem 2.1. *For any extensional and approximable K-model D, the following are equivalent:*

1. *D is hyperimmune,*
2. *D respects the quasi-approximation property,*
3. *D is inequationally fully abstract for* Λ*,*
4. *D is fully abstract for* Λ*.*

Proof.

- (1) $\Rightarrow$ (2): Theorem 2.48,
- (2) $\Rightarrow$ (3): inequational adequacy is the object of Theorem 2.49 and inequational completeness the one of Theorem 2.50,
- (3) $\Rightarrow$ (4): trivial,
- (4) $\Rightarrow$ (1): Theorem 2.51.

□

2.1. **Böhm trees.**

2.1.1. *Basic definitions.*

The Böhm trees provide one of the simplest semantics for the λ-calculus:

Definition 2.2. The set of *Böhm trees* is the co-inductive structure generated by the grammar:

$$\text{(Böhm trees)} \quad \boldsymbol{BT} \quad U, V \quad ::= \quad \Omega \quad | \quad \lambda x_1 ... x_n . y\ U_1 \cdots U_k \quad , \forall n, \forall k \geq 0$$

The *Böhm tree of a λ-term M* (*i.e.*, its interpretation), is defined by co-induction:

- If M head diverges, then $\boldsymbol{BT}(M) = \Omega$,
- if $M \to_h^* \lambda x_1 ... x_n . y\ N_1 \cdots N_k$, then
$$\boldsymbol{BT}(M) = \lambda x_1 ... x_n . y\ \boldsymbol{BT}(N_1) \cdots \boldsymbol{BT}(N_k).$$

Notice that a Böhm tree can be described as a finitely branching tree (of possibly infinite height) where nodes are labeled either by a constant Ω, or by a list of abstractions and by a head variable.

Capital final Latin letters U, V, W... will range over Böhm trees.

Example 2.3. The Böhm trees $\boldsymbol{BT}(\lambda x.x\,(\lambda y.x\,y))$, $\boldsymbol{BT}(x\,(\boldsymbol{I}\,\boldsymbol{I})\,(y\,(\boldsymbol{\Theta}\,\boldsymbol{I})))$, $\boldsymbol{\Theta}$ and $\boldsymbol{BT}(\Theta\,(\lambda uxy.y(u\,x))\,z)$ are described in Figure 3.

There exist Böhm trees that do not come from terms:

[5] Even if technically independent.

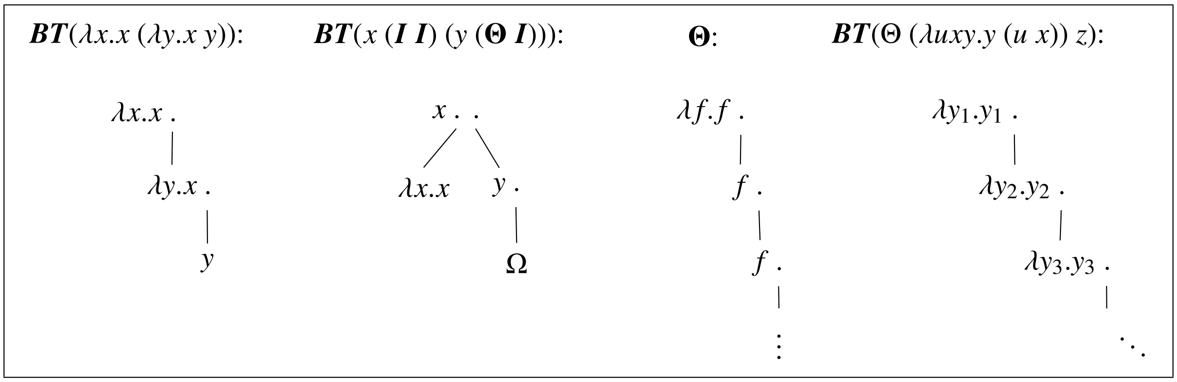
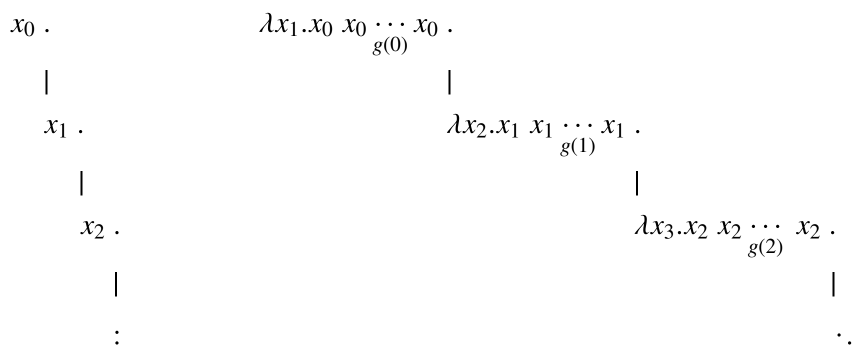


FIGURE 3. Some examples of Böhm trees.

Example 2.4. A Böhm tree with infinitely many free variables (such as the first one below) cannot be obtained from λ-terms that have finitely many free variables. Worse, if $g : \mathbb{N} \to \mathbb{N}$ is non recursive, then the second Böhm tree below does not come from any term (otherwise it would be possible to compute g from this term).



2.1.2. *Properties.*

The Böhm tree model model carries several interesting properties for the study of the untyped λ-calculus. By construction, it is sensible for the head reduction, and, moreover, it is adequate for $\mathcal{H}^*$ which is coarser. Moreover, those properties extend to inequations using the following natural notion of inclusion on Böhm trees:

Definition 2.5. The *inclusion of Böhm trees* $U \subseteq V$ is co-inductively defined by:

- $\Omega \subseteq V$ for all V
- If for all $i \leq k$, $U_i \subseteq V_i$, then

$$(\lambda x_1...x_n.y\ U_1 \cdots U_k)\ \subseteq (\lambda x_1...x_n.y\ V_1 \cdots V_k).$$

For readability, we will write $M \subseteq_{\boldsymbol{BT}} N$ whenever $\boldsymbol{BT}(M) \subseteq \boldsymbol{BT}(N)$.

The lower bounds of a Böhm tree U are obtained by replacing (possibly infinitely many) subtrees of U by Ω.

Example 2.6. For any M, we have the inclusion

$$\boldsymbol{\Theta}\ (\lambda uxy.x\ (u\ y)\ \boldsymbol{\Omega}) \quad \subseteq_{\boldsymbol{BT}} \quad \boldsymbol{\Theta}\ (\lambda uxy.x\ (u\ y)\ (M\ x))$$

$$
\begin{array}{lcl}
\lambda x_0 x_1.x_0\ \cdot\ \cdot & & \\
\quad | & \diagdown & \\
\lambda x_2.x_1\ \cdot\ \cdot & & \Omega \\
\quad | & \diagdown & \\
\lambda x_3.x_2\ \cdot\ \cdot & & \Omega \\
\quad | & \diagdown & \\
\quad \vdots & & \Omega
\end{array}
\quad \subseteq \quad
\begin{array}{lcl}
\lambda x_0 x_1.x_0\ \cdot\ \cdot & & \\
\quad | & \diagdown & \\
\lambda x_2.x_1\ \cdot\ \cdot & & \boldsymbol{BT}(M\ x_1) \\
\quad | & \diagdown & \\
\lambda x_3.x_2\ \cdot\ \cdot & & \boldsymbol{BT}(M\ x_2) \\
\quad | & \diagdown & \\
\quad \vdots\ \ddots & & \boldsymbol{BT}(M\ x_3)
\end{array}
$$

Proposition 2.7 ([4, Proposition 16.4.7])**.** *Böhm trees are inequationally adequate for $\mathcal{H}^*$,* i.e.

$$\textit{if} \qquad M \subseteq_{\boldsymbol{BT}} N \qquad\qquad \textit{then} \qquad M \sqsubseteq_{\mathcal{H}^*} N$$

The converse does not hold (because $\subseteq_{\boldsymbol{BT}}$ is not extensional), so that we do not have full abstraction, but rather a new (inequational) λ-theory called $\mathcal{BT}$.

Forcefully adding the extensionality in $\mathcal{BT}$, we obtain the theory $\mathcal{BT}\eta$ which is different from $\mathcal{H}^*$:

Example 2.8. The term $\boldsymbol{J} = \Theta\ (\lambda uxy.x\ (u\ y))$ defines the following Böhm tree:

$$
\begin{array}{llll}
\lambda x_0 x_1.x_0\ \cdot & & & \\
\qquad\quad | & & & \\
 & \lambda x_2.x_1\ \cdot & & \\
 & \qquad\ | & & \\
 & & \lambda x_3.x_2\ \cdot & \\
 & & & \ddots
\end{array}
$$

The behavior of this term is the same as the identity, so that we have $\boldsymbol{J} \equiv_{\mathcal{H}^*} \boldsymbol{I}$, but their Böhm trees are distinct and they are not η-convertible, so that $\boldsymbol{J} \not\equiv_{\mathcal{BT}\eta} \boldsymbol{I}$.

2.1.3. *Böhm trees and full abstraction.*

We have seen that $\boldsymbol{BT}$ is not fully abstract for $\mathcal{H}^*$ since it is not extensional; however, there are refinements using the notion of infinite η expansion that permit to say something about the full abstraction (Proposition 2.13).

Definition 2.9. We write by $\geq_\eta$ the *η-reduction on Böhm trees*, that is $U \geq_\eta V$ if $U = V = \Omega$ or if

$$
\begin{array}{rl}
 & U = \lambda x_1 ... x_{n+m}.y\ V_1 \cdots V_k\ x_{n+1} \cdots x_{n+m} \\
\text{and} & V = \lambda x_1 ... x_n. \quad y\ V_1 \cdots V_k
\end{array}
$$

where $x_{n+1}, ..., x_{n+m} \notin \mathrm{FV}(V_1, ..., V_k)$.

Definition 2.10. We write by $\geq_{\eta\infty}$ the co-inductive version of $\geq_\eta$, that is the coinductive relation generated by:

$$
\frac{}{\Omega \geq_\eta \Omega}\ (\eta\infty\omega) \qquad\qquad \frac{\forall i \leq k,\ U_i \geq_{\eta\infty} V_i \qquad \forall i \leq m,\ U_{k+i} \geq_{\eta\infty} x_{n+i}}{\lambda x_1 ... x_{n+m}.y\ U_1 \cdots U_{k+m} \geq_{\eta\infty} \lambda x_1 ... x_n.y\ V_1 \cdots V_k}\ (\eta\infty@)
$$

By abuse of notations, given two λ-terms M and N, we say that M *infinitely η-expands* N, written $M \geq_{\eta\infty} N$, if $\boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$.

Example 2.11. We have the inequations:

$$\begin{array}{ccccccccc}
\boldsymbol{BT}(\boldsymbol{I}) & \leq_{\eta\infty} & \boldsymbol{BT}(\boldsymbol{J}) & \leq_{\eta\infty} & \multicolumn{5}{c}{\boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda uxyz.x\ (u\ y)\ (u\ z)))} \\
\lambda x_0.x_0 & & \lambda x_0x_1.x_0\ . & & \lambda x_0x_1y_1.x_0\ .\ . & & & & \\
 & & | & & | & \diagdown & & & \\
 & \leq_{\eta\infty} & \lambda x_2.x_1\ . & \leq_{\eta\infty} & \lambda x_2y_2.x_1\ .\ . & & \lambda y_2z_2.y_1\ .\ . & & \\
 & & | & & | & \diagdown & & \diagdown & \\
 & & \lambda x_3.x_2\ . & & \lambda x_3y_3.x_2\ .\ . & & \lambda y_3z_3.y_2\ .\ . & & \ddots \\
 & & | & & | & \diagdown & & \diagdown & \\
 & & \vdots & & \ddots & & \ddots & & \ddots
\end{array}$$

Remark 2.12. The η-reduction on Böhm trees is not directly related to the η-reduction on λ-terms.

For example

$$\boldsymbol{\Theta}\ (\lambda uzx.x\ (y\ z)) \not\leq_\eta \lambda x.\boldsymbol{\Theta}\ (\lambda uzx.x\ (yz))\ x.$$

Since x is not free, however, this reduction holds at the level of Böhm trees.

Conversely, we have

$$\boldsymbol{\Theta}\ (\lambda uz.z\ (u\ z)) \leq_\eta \boldsymbol{\Theta}\ (\lambda uzx.z\ (u\ z)\ x)$$

even while the Böhm trees are fairly different.

However, the η-reduction on λ-terms is directly implied by the infinite η reduction.

Using $\leq_{\eta\infty}$, we can characterize the notion of observational equivalence (*i.e.*, $\mathcal{H}^*$)

Proposition 2.13 ([4, Theorem 19.2.9])**.** *For any terms $M, N \in \Lambda$, $M \sqsubseteq_{\mathcal{H}^*} N$ iff there exist two Böhm trees U, V such that:*

$$\boldsymbol{BT}(M) \leq_{\eta_\infty} U \subseteq V \geq_{\eta\infty} \boldsymbol{BT}(N).$$

Example 2.14. In $\mathcal{H}^*$, we have the equivalence:

$$\begin{array}{ccccccc}
\boldsymbol{J} & & \multicolumn{1}{c}{\equiv_{\mathcal{H}^*}} & \multicolumn{4}{c}{\boldsymbol{\Theta}\ (\lambda uxyz.x\ y\ (u\ z))} \\
\lambda x_0x_1.x_0\ . & & \lambda x_0x_1y_1.x_0\ .\ . & & & & \lambda x_0x_1y_1.x_0\ x_1\ . \\
| & & | & \diagdown & & & | \\
\lambda x_2.x_1\ . & \leq_{\eta\infty} & \lambda x_2.x_1\ . & & \lambda x_2y_2.y_1\ x_2\ . & \geq_{\eta\infty} & \lambda x_2y_2.y_1\ x_2\ . \\
| & & | & & | & & | \\
\lambda x_3.x_2\ . & & \lambda x_3.x_2\ . & & \lambda x_3y_3.y_2\ x_3 & & \lambda x_3y_3.y_2\ x_3\ . \\
| & & | & & | & & | \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The following trivial corollary will be rather useful for proving observational equivalences:

Corollary 2.15. *For all $M, N \in \Lambda$,*

$$M \geq_{\eta\infty} N \Rightarrow M \equiv_{\mathcal{H}^*} N.$$

Proof. By Proposition 2.13 and since $\boldsymbol{BT}(M) \leq_{\eta\infty} \boldsymbol{BT}(M) \subseteq \boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$. □

2.1.4. *Subclasses of Böhm trees.*

Before saying anything on interpretation of Böhm trees in a K-model, we define some subclasses of Böhm trees that will work as potential bases. Such bases can be used to interpret a Böhm tree in our models as the sup of the interpretations of its approximants.[6]

The only base that appears in the literature is the class $\boldsymbol{BT_f}$ of finite Böhm trees. However, we will oppose it the larger classes $\boldsymbol{BT_{\Omega f}}$ and $\boldsymbol{BT_{qf}}$ of Ω-finite and quasi-finite Böhm trees. The Ω-finiteness when applied to an approximant of an actual term (via its translation into a Böhm tree) is a property that insure the recursivity of the tree (Lemma. 2.18). The quasi-finite Böhm trees are the Ω-finite Böhm trees that are somehow "stable" with respect to $\leq_{\eta\infty}$ and $\geq_{\eta\infty}$ (Lemma. 2.21).

Definition 2.16. We define the following classes over Böhm trees:

- The set of *finite Böhm tree*s, denoted $\boldsymbol{BT_f}$, is the set of Böhm trees inductively generated by the grammar of Definition 2.2 (or equivalently Böhm trees of finite height). Given a term M, we denote $\boldsymbol{BT_f}(M)$ the set of finite Böhm trees U such that $U \subseteq \boldsymbol{BT}(M)$.
- The set of Ω-*finite Böhm tree*s, denoted $\boldsymbol{BT_{\Omega f}}$, is the set of Böhm trees that contain a finite number of occurrences of Ω.
- The set of *quasi-finite Böhm tree*, denoted $\boldsymbol{BT_{qf}}$, is the set of those Ω-finite Böhm trees having their number of occurences of each (free and bounded) variables recursively bounded. Formally, there is a recursive function g such that variables abstracted at depth[7]n cannot occur at depth greater than $g(n)$.

Capital final Latin letters X, Y, Z... will range over any of those classes of Böhm trees. We will use the notation $\subseteq_f$ (resp. $\subseteq_{\Omega f}$ and $\subseteq_{qf}$) for the inclusion restricted to $\boldsymbol{BT_f} \times \boldsymbol{BT}$ (resp. $\boldsymbol{BT_{\Omega f}} \times \boldsymbol{BT}$ and $\boldsymbol{BT_{qf}} \times \boldsymbol{BT}$).

In particular, to any finite Böhm tree U corresponds a term M obtained by replacing every symbol Ω by the diverging term $\boldsymbol{\Omega}$. By abuse of notation, we may use one instead of the other.

Example 2.17. The identity $\boldsymbol{I}$ corresponds to a finite Böhm tree and thus is in all three classes. The term $\lambda z.\boldsymbol{\Theta}\ (\lambda ux.z\ u)$ has a Böhm tree that is Ω-finite but not quasi-finite. The term $\boldsymbol{\Theta}\ (\lambda ux.x\ u\ \Omega)$ has a Böhm tree that is neither of these classes.

$$\boldsymbol{BT}(\lambda z.\boldsymbol{\Theta}\ (\lambda ux.z\ u)) = \begin{array}{r} \lambda z x_1.z\ . \\ | \\ \lambda x_2.z\ . \\ | \\ \lambda x_3.z\ . \\ \vdots \end{array} \qquad \boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda ux.x\ u\ \Omega)) = \begin{array}{r} \lambda x_1.x_1\ .\ \Omega \\ |\quad \\ \lambda x_2.x_2\ .\ \Omega \\ |\quad \\ \lambda x_3.x_3\ .\ \Omega \\ \vdots\quad \end{array}$$

Lemma 2.18. For all terms M, if $X \in \boldsymbol{BT_{\Omega f}}$ and $X \subseteq \boldsymbol{BT}(M)$, then X is a recursive Böhm tree.

Proof. First remark that only X has to be recursive, not the proof of $X \subseteq \boldsymbol{BT}(M)$. Moreover, we only have to show that there exists a recursive construction of X, we do not have to generate it constructively.

[6]We will see that as a coinductive structure, a Böhm trees may have several possible interpretations into a given model.

[7]We consider that free variables are "abstracted" at depth 0.

There is a finite number of Ω's in X whose positions $p \in P$ can be guessed beforehand by an oracle that is finite thus recursive. After that, it suffices to compute the Böhm tree of M except in these positions where we directly put an Ω. This way the program is always productive as any Ω of M (*i.e.*, any non terminating part of the process of computation of $\boldsymbol{BT}(M)$) will be shaded by a guessed Ω of X (potentially far above). $\square$

Lemma 2.19. Let $U, V \in \boldsymbol{BT}$. If $U \leq_{\eta\infty} V$ (def. 2.10), there is a bijection between the Ω's in U and those in V.

Proof. Recall that $U \leq_{\eta\infty} V$ is the relation whose proofs range over the coinductive sequents generated by

$$\frac{}{\Omega \geq_\eta \Omega}\ (\eta\infty\omega) \qquad\qquad \frac{\forall i \leq k,\ U_i \geq_{\eta\infty} V_i \qquad \forall i \leq m,\ U_{k+i} \geq_{\eta\infty} x_{n+i}}{\lambda x_1 ... x_{n+m}.y\ U_1 \cdots U_{k+m} \geq_{\eta\infty} \lambda x_1 ... x_n.y\ V_1 \cdots V_k}\ (\eta\infty @)$$

Remark that this system is deterministic so that a sequent $U \leq_{\eta\infty} V$ has at most one proof. In particular the occurrences of rule $(\eta\infty\omega)$ describe the pursued bijection. $\square$

Lemma 2.20. For all $U, V \in \boldsymbol{BT}$ such that $U \leq_{\eta\infty} V$, $U \in \boldsymbol{BT}_{qf}$ iff $V \in \boldsymbol{BT}_{qf}$.

Proof. By Lemma 2.19, we know that $U \in \boldsymbol{BT}_{\Omega f}$ iff $V \in \boldsymbol{BT}_{\Omega f}$.

It is easy to see that if variable occurrences are bounded by g in U, then they will be bounded by $(n \mapsto max(g(n), 1))$ in V and conversly. Indeed an $\eta\infty$-expansion/reduction will not change the depth of any variable, and will only delete/introduce abstraction whose variable will be used exactly once at depth 1. $\square$

Lemma 2.21. Both ordering $\leq_{\eta\infty}$ and $\geq_{\eta\infty}$ distribute over $\subseteq_{qf}$, and the ordering $\geq_{\eta\infty}$ distributes over $\subseteq_f$:

- For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT}_{qf}$ such that $X \subseteq_{qf} U \leq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT}_{qf}$ such that[8]
$$\begin{array}{ccc} U & \leq_{\eta\infty} & V \\ \cup\!|_{qf} & \rightsquigarrow & \cup\!|_{qf} \\ X & \leq_{\eta\infty} & Y. \end{array}$$
- For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT}_{qf}$ such that $X \subseteq_{qf} U \geq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT}_{qf}$ such that
$$\begin{array}{ccc} U & \geq_{\eta\infty} & V \\ \cup\!|_{qf} & \rightsquigarrow & \cup\!|_{qf} \\ X & \geq_{\eta\infty} & Y. \end{array}$$
- For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT}_f$ such that $X \subseteq_f U \geq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT}_f$ such that
$$\begin{array}{ccc} U & \geq_{\eta\infty} & V \\ \cup\!|_{f} & \rightsquigarrow & \cup\!|_{f} \\ X & \geq_{\eta\infty} & Y. \end{array}$$

Proof.

- Distribution of $\leq_{\eta\infty}$ over $\subseteq_{qf}$:
  We create $Y \in \boldsymbol{BT}$ such that $X \leq_{\eta\infty} Y \subseteq V$ by co-induction (remark that, by Lemma 2.20, we obtain $V \in \boldsymbol{BT}_{qf}$):
  - $X = \Omega$: put $Y = \Omega$.

[8]This is a commuting diagram, the $\rightsquigarrow$ arrow only recalls that Y is obtained from X, U and V.

  - Otherwise: we have

$$X = \lambda x_1...x_n.y\, X_1 \cdots X_m, \qquad U = \lambda x_1...x_n.y\, U_1 \cdots U_m, \qquad V = \lambda x_1...x_{n+k}.y\, V_1 \cdots V_{m+k},$$

    such that $X_i \subseteq_{qf} U_i \preceq_{\eta\infty} V_i$ for $i \leq m$ and $x_{n+i} \preceq_{\eta\infty} V_{m+i}$ (thus $V_{m+i} \in \boldsymbol{BT_{qf}}$) for $i \leq k$. By co-induction hypothesis we have $(Y_i)_{i\leq m}$ such that $X_i \preceq_{\eta\infty} Y_i \subseteq V_i$ for $i \leq m$, we thus set

$$Y = \lambda x_1...x_{n+k}.y\, Y_1 \cdots Y_m\, V_{m+1} \cdots V_{m+k}.$$

- Distribution of $\succeq_{\eta\infty}$ over $\subseteq_{qf}$:
  We create $Y \in \boldsymbol{BT}$ such that $X \succeq_{\eta\infty} Y \subseteq V$ by co-induction, then, by Lemma 2.20, we obtain that $V \in \boldsymbol{BT_{qf}}$:
  - $X = \Omega$: put $Y = \Omega$.
  - Otherwise: we have

$$X = \lambda x_1...x_{n+k}.y\, X_1 \cdots X_{m+k}, \qquad U = \lambda x_1...x_{n+k}.y\, U_1 \cdots U_{m+k}, \qquad V = \lambda x_1...x_n.y\, V_1 \cdots V_m,$$

    such that $X_i \subseteq_{qf} U_i \succeq_{\eta\infty} V_i$ for $i \leq m$ and $X_{m+i} \subseteq_{qf} U_{m+i} \succeq_{\eta\infty} x_{n+i}$ for $i \leq k$. By co-induction hypothesis we have $(Y_i)_{i\leq m+k}$ such that $X_i \succeq_{\eta\infty} Y_i \subseteq V_i$ for $i \leq m$, and $X_{m+i} \succeq_{\eta\infty} Y_{m+i} \subseteq x_{n+i}$ for $i \leq k$; we thus set

$$Y = \lambda x_1...x_{n+k}.y\, Y_1 \cdots Y_m.$$

- Distribution of $\succeq_{\eta\infty}$ over $\subseteq_f$:
  We create $Y \in \boldsymbol{BT_f}$ similarly to the previous case except that we proceed by induction on X:
  - $X = \Omega$: put $Y = \Omega$.
  - Otherwise: we have

$$X = \lambda x_1...x_{n+k}.y\, X_1 \cdots X_{m+k}, \qquad U = \lambda x_1...x_{n+k}.y\, U_1 \cdots U_{m+k}, \qquad V = \lambda x_1...x_n.y\, V_1 \cdots V_m,$$

    such that $X_i \subseteq_f U_i \succeq_{\eta\infty} V_i$ for $i \leq m$ and $X_{m+i} \subseteq_f U_{m+i} \succeq_{\eta\infty} x_{n+i}$ for $i \leq k$. By co-induction hypothesis we have $(Y_i)_{i\leq m+k}$ such that $X_i \succeq_{\eta\infty} Y_i \subseteq_f V_i$ for $i \leq m$, and $X_{m+i} \succeq_{\eta\infty} Y_{m+i} \subseteq_f x_{n+i}$ for $i \leq k$; we thus set

$$Y = \lambda x_1...x_{n+k}.y\, Y_1 \cdots Y_m.$$

□

2.1.5. *Interpretations of Böhm trees.*

Böhm trees can be seen as normal forms of infinite depth. As such, one can define an interpretation of Böhm trees in a model via fixponts. However, there is no *a priori* reason to choose one specific fixpoint. We will formalize the notion of interpretation of Böhm trees in Definition 2.22. Then, using the description of such fixpoints, we will see in Propsition 2.24 that the set of interpretations forms a complete lattice.

The minimal interpretation, called the inductive interpretation (Def. 2.25), is the canonical choice and has been used often in the literature to describe the approximation property (Def. 2.27). Roughly speaking, the approximation property states the coherence of the interpretation of terms and the inductive interpretation of Böhm trees.

The complete lattice of interpretations is richer than the sole inductive interpretation. Another canonical interpretation is the maximal one, called co-inductive interpretation (Def. 2.25). Unfortunately, no equivalent version of approximation property can be given for the co-inductive interpretation (more exactly, no K-model can satisfy it).

However, we can look for an interpretation that is both, as large as possible and with a useful notion of coherence with the λ-calculus. We found the quasi-finite interpretation (Def. 2.31) that is basically the minimal interpretation whose restriction to quasi-finite Böhm trees corresponds to the co-inductive interpretation. The property stating the coherence of interpretations is the quasi-approximation property (Def. 2.32). We will see later on that, in the presence of the approximation property and extensionality, the quasi-approximation property is equivalent to hyperimmunity and to full abstraction for $\mathcal{H}^*$.

Definition 2.22. Let D be a K-model. We call *proto-interpretation of Böhm trees* any total function $[\![-]\!]_*$ that maps elements $U \in \boldsymbol{BT}$ to initial segments of $D^{\mathrm{FV}(U)} \Rightarrow D$ (where $\mathrm{FV}(U)$ denotes the free variables of U).

An *interpretation of Böhm trees* is a proto-interpretation $[\![.]\!]_*$ respecting the following:

- The interpretation of Ω is always empty:
$$[\![\Omega]\!]_*^{\vec{x}} = \emptyset.$$
- The interpretation of an abstraction $\lambda y.U$ satisfies:
$$[\![\lambda y.U]\!]_*^{\vec{x}} = \{(\vec{a}, b{\to}\alpha) \mid (\vec{a}b, \alpha) \in [\![U]\!]_*^{\vec{x}y}\}.$$
- The interpretation of a list of applications $x_i\ U_1 \cdots U_n$ (for $n \geq 0$), satisfies:
$$[\![x_i\ U_1 \cdots U_n]\!]_*^{\vec{x}} = \{(\vec{a}, \alpha) \mid \exists b_1{\to}\cdots{\to}b_n{\to}\alpha \leq \alpha' \in a_i, \forall j \leq n, \forall \beta \in b_j, (\vec{a}, \beta) \in [\![U_j]\!]_*^{\vec{x}}\}$$

Remark 2.23. The different interpretations coincide on finite Böhm trees, thus we can write $[\![X]\!]^{\vec{x}}$ for any $X \in \boldsymbol{BT_f}$ without ambiguity, independently of the interpretation. Moreover, if the model is sensible, $[\![X]\!]^{\vec{x}}$ is the same as the interpretation of X considered as a λ-term (by replacing occurrences of Ω by the diverging term $\boldsymbol{\Omega}$).

The interpretations differ on the infinite Böhm trees. Fortunately, the set of interpretations forms a complete lattice.

Proposition 2.24. *The poset of interpretations (with pointwise inclusion) is a complete lattice.*

Proof. We show that the set of the interpretation is the set of the fixpoints of a Scott-continuous function ζ on the complete lattice of proto-interpretations (with pointwise order).

The function ζ maps a proto-interpretation $[\![.]\!]_*$ to the proto-interpretation $[\![.]\!]_{\zeta(*)}$ defined as follows:

- The interpretation of Ω is always empty:
$$[\![\Omega]\!]_{\zeta(*)} = \emptyset.$$
- The interpretation of $\lambda y.U$ is the same as for λ-terms:
$$[\![\lambda y.U]\!]_{\zeta(*)}^{\vec{x}} = \{(\vec{a}, b{\to}\alpha) \mid (\vec{a}b, \alpha) \in [\![U]\!]_*^{\vec{x}y}\}.$$
- The interpretation of $x_i\ U_1 \cdots U_n$ satisfies:
$$[\![x_i\ U_1 \cdots U_n]\!]_{\zeta(*)}^{\vec{x}} = \{(\vec{a}, \alpha) \mid \exists b_1{\to}\cdots{\to}b_n{\to}\alpha \leq \alpha' \in a_i, \forall j \leq n, \forall \beta \in b_j, (\vec{a}, \beta) \in [\![U_j]\!]_*^{\vec{x}}\},$$

The two first equations trivialy preserve any sup. And the third equation preserves the directed sup since all b_j are finite. These three equations preserve the directed sups, so that ζ is continuous. It is folklore that the set of fixpoints of a Scott-continuous function form a complete lattice. $\square$

$$\frac{\Gamma, x : a \vdash U : \alpha}{\Gamma \vdash \lambda x.U : a \to \alpha}\ (BT\text{-}\lambda)$$

$$\frac{b_1 \to \cdots \to b_n \to \beta \in a \qquad \alpha \leq \beta \qquad \forall i \leq n, \forall \gamma \in b_i,\ \Gamma, x : a \vdash U_i : \gamma}{\Gamma, x : a \vdash x\, U_1 \cdots U_n : \alpha}\ (BT\text{-}@)$$

FIGURE 4. Intersection type system for Böhm trees. Notice that the intersection is hiddent in the membership condition in the first premise of (BT-@).

Definition 2.25. The minimal interpretation is the *inductive interpretation*

$$[\![U]\!]^{\vec{x}}_{ind} = \bigcup_{\substack{X \subseteq U \\ X \in \boldsymbol{BT_f}}} [\![X]\!]^{\vec{x}}.$$

The maximal interpretation is called the *co-inductive interpretation* and denoted $[\![.]\!]^{\vec{x}}_{coind}$.

The idea of intersection types can be generalized to Böhm trees. We introduce in Figure 4 the corresponding intersection type system. There is no rule for Ω since it has an empty interpretation. Remark, moreover, that the rule (BT-@) seems complicated, but is just the aggregation of rules (I-id), (I-$weak$), (I- $\leq$) and (I-@) of Figure 2. The difference between the inductive and the co-inductive interpretations lies on the finiteness of the allowed derivations in this system.

Proposition 2.26. *Let U be a Böhm tree, then:*

- $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}}_{ind}$ *iff the type judgment* $\vec{x} : \vec{a} \vdash U : \alpha$ *has a finite derivation using the rules of Figure 4.*
- $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}}_{coind}$ *iff the type judgment* $\vec{x} : \vec{a} \vdash U : \alpha$ *has a possibly infinite derivation using the rules of Figure 4.*

Definition 2.27. We say that D respects the *approximation property*, or that D is *approximable*, if the interpretation of any term corresponds to the inductive interpretation of its Böhm tree, *i.e.* if the following diagram commutes:

$$\begin{tikzcd} \Lambda \arrow[rr, "{[\![.]\!]}"] \arrow[dr, "\boldsymbol{BT}(.)"'] & & D \\ & \boldsymbol{BT} \arrow[ur, "{[\![.]\!]_{ind}}"'] & \end{tikzcd}$$

Lemma 2.28. If D is extensional and approximable, and if M and N are two terms such that $M \geq_{\eta\infty} N$ (def. 2.10), then $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$.

Proof. Let $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$, by the approximation property there is a finite $U \subseteq_f \boldsymbol{BT}(M)$ such that $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}}$. Since $U \subseteq_f \boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$, we can apply Lemma 2.21 to find $V \in \boldsymbol{BT_f}$ such that $U \geq_{\eta\infty} V \subseteq_f \boldsymbol{BT}(N)$. However, between finite Böhm trees, an $\infty\eta$-expansion is a usual η-expansion, so that $U \geq_\eta V \subseteq_f \boldsymbol{BT}(N)$. We thus have (using extensionality), $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}} = [\![V]\!]^{\vec{x}} \subseteq [\![M]\!]^{\vec{x}}$ because the model is extensional. □

The approximation property is a common condition enjoyed by all known K-models.[9]

Example 2.29. All the K-models of Example 1.15 except P_∞ (that is not even sensible) are approximable, regardless of them being fully abstract or not.

Our goal is to modify our set of approximants so that we could characterize the full abstraction.

[9]Provided that they equalize terms with the same Böhm trees (which is a necessary condition for full abstraction).

Remark 2.30. A vain attempt would consist on replacing the inductive interpretation (in the definition of the approximation property) by the co-inductive one. The diagram of Definition 2.27 would never commute:

For any sensible K-model and any $\alpha \in D$, if $M = \mathbf{\Theta}\ (\lambda u.z\ u)$, then

$$(\{\{\alpha\}\to\alpha\}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^z_{coind} \qquad\qquad (\{\{\alpha\}\to\alpha\}, \alpha) \notin [\![M]\!]^z.$$

Indeed, if $(\{\{\alpha\}\to\alpha\}, \alpha) \in [\![M]\!]^z$ it would give $\alpha \in [\![M[\boldsymbol{I}/z]]\!] = [\![\mathbf{\Theta}\ \boldsymbol{I}]\!] = \emptyset$. Moreover, since $\boldsymbol{BT}(M) = z\ \boldsymbol{BT}(M)$, we co-inductively get that $(\{\{\alpha\}\to\alpha\}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^z_{coind}$.

In this example, the co-inductive interpretation of $\boldsymbol{BT}(\mathbf{\Theta}\ (\lambda ux.z\ u))$ is incoherent with the term interpretation because it uses the z infinitely often.[10] In order to get rid of this incoherence we can use a guarded fixpoint.

In order to recover a meaningful property, we will use the *quasi-finite interpretation*. This is the least interpretation whose restriction to quasi-finite Böhm trees is the co-inductive interpretation.

Definition 2.31. The *quasi-finite interpretation of Böhm trees* is defined by

$$[\![U]\!]^{\vec{x}}_{qf} = \bigcup_{\substack{X\subseteq U\\ X\in \boldsymbol{BT}_{qf}}} [\![X]\!]^{\vec{x}}_{coind}.$$

Definition 2.32. We say that D respects the *quasi-approximation property*, or is *quasi-approximable*, if the interpretation of any term corresponds to the quasi-finite interpretation of its Böhm tree, *i.e.* if the following diagram commutes:

$$\begin{tikzcd} \Lambda \arrow[rr, "{[\![.]\!]}"] \arrow[dr, "\boldsymbol{BT}(.)"'] & & D \\ & \boldsymbol{BT} \arrow[ur, "{[\![.]\!]_{qf}}"'] & \end{tikzcd}$$

Example 2.33. We will prove that the quasi-approximation property is equivalent to hyperimmunity and full abstraction for $\mathcal{H}^*$ (in presence of approximation property and extensionality). So models that are hyperimmune, like D_∞, respect it and those that are not, like D^*_∞, do not. In the case of D^*_∞, for example, the quasi-approximation property is refuted by $\boldsymbol{J}$, indeed $p \in [\![\boldsymbol{BT}(\boldsymbol{J})]\!]_{qf} - [\![\boldsymbol{J}]\!]$.

Remark 2.34. Notice that in general, approximability and quasi-approximability are independent (in the sense that none implies the other).

2.1.6. *Technical lemma.*

This section shows that the relation $\leq_{\eta\infty}$ in $\boldsymbol{BT}$ is pushed along the co-inductive interpretation into equality at the level of the model. This property will be useful as it generalizes easily to the quasi-finite interpretation.

Lemma 2.35. Let D be an extensional K-model and let U, V be two Böhm trees such that $U \leq_{\eta\infty} V$. Then $[\![U]\!]^{\vec{x}}_{coind} = [\![V]\!]^{\vec{x}}_{coind}$.

Proof. We will prove separately the two inclusions.

[10] Notice that in a relational model [17] this issue would not hold (even if other problems would come later) since in any elements of the interpretation $(a, \alpha) \in [\![\lambda x.M]\!]$ the a is a finite multiset which can only "see" a finite number occurences of z.

- We will show that the proto-interpretation $[\![V]\!]_*^{\vec{x}} = \bigcup_{U \geq_{\eta\infty} V} [\![U]\!]_{coind}^{\vec{x}}$ over Böhm trees is an interpretation. This is sufficient since, $[\![\_]\!]_{coind}$ being the greatest interpretation, we will have
$$[\![V]\!]_{coind} \subseteq \bigcup_{U \geq_{\eta\infty} V} [\![U]\!]_{coind}^{\vec{x}} = [\![V]\!]_* \subseteq [\![V]\!]_{coind}.$$
  - Interpretation over Ω:
$$[\![\Omega]\!]_*^{\vec{x}} = \bigcup_{U \geq_{\eta\infty} \Omega} [\![U]\!]_{coind}^{\vec{x}} = [\![\Omega]\!]_{coind}^{\vec{x}} = \emptyset.$$
  - Otherwise:
$$\begin{aligned}
&[\![\lambda x_{n+1} \ldots x_s . x_j \; V_1 \cdots V_k]\!]_*^{\vec{x}} \\
&\quad = \bigcup_{U \geq_{\eta\infty} \lambda x_{n+1} \ldots x_s . x_j \; V_1 \cdots V_k} [\![U]\!]_{coind}^{\vec{x}} \\
&\quad = \bigcup_m \bigcup_{(U_i \geq_{\eta\infty} V_i)_{i \leq k}} \bigcup_{U_{k+i} \geq_{\eta\infty} x_{s+i}} [\![\lambda x_{n+1} \ldots x_{s+m} . x_j \; U_1 \cdots U_{k+m}]\!]_{coind}^{\vec{x}} \\
&\quad = \bigcup_m \bigcup_{U_i \geq_{\eta\infty} V_i} \bigcup_{U_{k+i} \geq_{\eta\infty} x_{s+i}} \{((a_i)_{i \leq n}, a_{n+1} \to \cdots a_{s+m} \to \alpha) \mid \exists c_1 \to \cdots \to c_{k+m} \to \alpha \leq \alpha' \in a_j, \\
&\qquad\qquad \forall t \leq k{+}m, \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![U_t]\!]_{coind}^{\vec{x}^{s+m}}\} \\
&\quad = \bigcup_m \{((a_i)_{i \leq n}, a_{n+1} \to \cdots a_{s+m} \to \alpha) \ \mid \exists c_1 \to \cdots \to c_{k+m} \to \alpha \leq \alpha' \in a_j, \\
&\qquad\qquad \forall t \leq k, \forall \beta \in c_t,\ (\vec{a}, \beta) \in \bigcup_{U_t \geq_{\eta\infty} V_t} [\![U_t]\!]_{coind}^{\vec{x}^{s+m}} \\
&\qquad\qquad \forall t \leq m, \forall \beta \in c_{k+t},\ (\vec{a}, \beta) \in \bigcup_{U_{k+t} \geq_{\eta\infty} x_{s+t}} [\![U_t]\!]_{coind}^{\vec{x}^{s+m}}\} \\
&\quad = \bigcup_m \{((a_i)_{i \leq n}, a_{n+1} \to \cdots a_{s+m} \to \alpha) \ \mid \exists c_1 \to \cdots \to c_{k+m} \to \alpha \leq \alpha' \in a_j, \\
&\qquad\qquad \forall t \leq k, \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]_*^{\vec{x}^{s+m}} \\
&\qquad\qquad \forall t \leq m, \forall \beta \in c_{k+t},\ (\vec{a}, \beta) \in [\![x_{s+t}]\!]_*^{\vec{x}^{s+m}}\} \\
&\quad = \bigcup_m \{((a_i)_{i \leq n}, a_{n+1} \to \cdots a_{s+m} \to \alpha) \ \mid \exists c_1 \to \cdots \to c_{k+m} \to \alpha \leq \alpha' \in a_j, \\
&\qquad\qquad \forall t \leq k, \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]_*^{\vec{x}^{s+m}}\}
\end{aligned}$$
  This proves that if $U \geq_{\eta\infty} V$, then $[\![U]\!]_{coind}^{\vec{x}} \subseteq [\![V]\!]_*^{\vec{x}} \subseteq [\![V]\!]_{coind}^{\vec{x}}$.
- To prove the converse, it is sufficient to show that the proto-interpretation $[\![V]\!]_* = \bigcup_{U \leq_{\eta\infty} V} [\![V]\!]_{coind}$ is an interpretation:
  - Interpretation over Ω:
$$[\![\Omega]\!]_*^{\vec{x}} = \bigcup_{U \leq_{\eta\infty} \Omega} [\![U]\!]_{coind}^{\vec{x}} = [\![\Omega]\!]_{coind}^{\vec{x}} = \emptyset.$$

- If $V_s \not\geq_{\eta\infty} x_k$ and $V_{s+i} \geq_{\eta\infty} x_{k+i}$ (for $1 \leq i \leq m$) and $j \leq k$:

$$
\begin{aligned}
&[\![\lambda x_{n+1}...x_{k+m}.x_j\ V_1 \cdots V_{s+m}]\!]_*^{\vec{x}^n}\\
&= \bigcup_{U \preceq_{\eta\infty} \lambda x_{n+1}...x_{k+m}.x_j\ V_1 \cdots V_{s+m}} [\![U]\!]_{coind}^{\vec{x}^n}\\
&= \bigcup_{m' \leq m} \bigcup_{U_t \preceq_{\eta\infty} V_t} [\![\lambda x_{n+1}...x_{k+m'}.x_j\ U_1 \cdots U_{s+m'}]\!]_{coind}^{\vec{x}^n}\\
&= \bigcup_{m' \leq m} \bigcup_{U_t \preceq_{\eta\infty} V_t} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m'}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in [\![U_t]\!]_{coind}^{\vec{x}^{k+m'}}\}\\
&= \bigcup_{m' \leq m} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m'}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in \bigcup_{U_t \preceq_{\eta\infty} V_t} [\![U_t]\!]_{coind}^{\vec{x}^{k+m'}}\}\\
&= \bigcup_{m' \leq m} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m'}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in [\![V_t]\!]_*^{\vec{x}^{k+m'}}\}\\
&= \bigcup_{m' \leq m} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to a_{k+m'+1}\to \cdots a_{n+m}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in [\![V_t]\!]_*^{\vec{x}^{k+m'}}\}\\
&= \bigcup_{m' \leq m} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to a_{k+m'+1}\to \cdots a_{k+m}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in [\![V_t]\!]_*^{\vec{x}^{k+m'}}\\
&\qquad \forall m' \leq t \leq m, \forall\beta \in a_{k+t}, (\vec{a},\beta) \in [\![x_{k+t}]\!]_*^{\vec{x}^{k+m'}}\}\\
&= \bigcup_{m' \leq m} \{(a_i)_{i\leq n}, a_{n+1}\to \cdots a_{k+m}\to\alpha) \mid \exists c_1\to \cdots \to c_{s+m'}\to a_{k+m'+1}\to \cdots a_{k+m}\to\alpha \leq \alpha' \in a_j,\\
&\qquad \forall t \leq s{+}m', \forall\beta \in c_t,\ (\vec{a},\beta) \in [\![V_t]\!]_*^{\vec{x}^{k+m'}}\\
&\qquad \forall m' \leq t \leq m, \forall\beta \in a_{k+t}, (\vec{a},\beta) \in [\![V_{s+t}]\!]_*^{\vec{x}^{k+m'}}\}
\end{aligned}
$$

This proves that if $U \preceq_{\eta\infty} V$, then $[\![U]\!]_{coind}^{\vec{x}} \subseteq [\![V]\!]_*^{\vec{x}} \subseteq [\![V]\!]_{coind}^{\vec{x}}$.

□

2.2. **Hyperimmunity implies full abstraction.**

In this section we will prove the step (1) ⇒ (2) of the main theorem (Th. 1.23). This will be done using the quasi-approximation property to decompose the proof into two steps. Indeed, we will see that in the presence of the approximation property, hyperimmunity implies the quasi-approximation property that itself implies the full abstraction for $\mathcal{H}^*$. Those two implications will be proved separately in Theorems 2.48 and 2.50.

2.2.1. *Hyperimmunity and approximation imply quasi-approximation.*

Firstly, we are introducing tree-hyperimmunity that is equivalent to hyperimmunity (Lemma 2.37). The reason to introduce this new formalism is quite simple. For the proof of Theorem 2.48, we will have to contradict hyperimmunity starting from a term M that contradicts quasi-approximability.

Recall that refuting hyperimmunity amounts to exhibiting a non-hyperimmune function (*i.e.*, bounded by a recursive function g) and a sequence $(\alpha_i)_i \in D^{\mathbb{N}}$ with a non well founded chain bounded by g (see Definition 1.19).

The refutation of quasi-approximability by M gives a recursive procedure that bounds the non-hyperimmune function g. However, the procedure does generally not directly construct the values of this function, but also performs a lot of useless computation; this is due to the refuting term M not being optimal. Thus, we will simply construct an infinite tree and use König lemma[11] to find an infinite branch that contradicts hyperimmunity.

Generalizing hyperimmunity from sequences to trees allows us to apply a well-known theorem of recursion theory. This theorem states the equivalence between hyperimmune functions and infinite paths in recursive $\mathbb{N}$-labeled trees.[12] That is why we can generalise hyperimmune functions to infinite recursive $\mathbb{N}$-labeled trees. The sequence $(\alpha_i)_i \in D^{\mathbb{N}}$, similarly, becomes a partial (but infinite) labeling of the recursive tree. The sequence has to be partial in order to select a specific hyperimmune path.

Definition 2.36. Let D be a K-model.
A $\mathbb{N}$-*labeled* tree T is a finitely branching tree where nodes are labeled by $\mathbb{N}$, we denote by $T(\mu)$ the $\mathbb{N}$-label of the node μ in T.
A *D-decoration* of a $\mathbb{N}$-labeled T is a partial function of infinite domain $\partial_D : T \to D$ such that for every couple of nodes ν and μ that are father and son in T, if $\mu \in dom(\partial_D)$, then $\nu \in dom(\partial_D)$ and:

$$\partial_D(\nu) = a_1 \to \cdots \to a_{T(\mu)} \to \alpha \qquad \Rightarrow \qquad \partial_D(\mu) \in a_{T(\mu)}.$$

A K-model D is *tree-hyperimmune* if none of the $\mathbb{N}$-labeled and D-decorated tree is recursive.

Lemma 2.37. A K-model D is tree-hyperimmune iff it is hyperimmune.

Proof.
- We assume that there is a recursive g and a sequence $(\alpha_n)_n$ refuting hyperimmunity. We define the tree T given by the set of nodes $\{\omega \in \mathbb{N}^* \mid \forall n \leq |\omega|, \omega_n \leq g(n)\}$ of finite sequences bounded by g and ordered by prefix; the $\mathbb{N}$-labeling is given by $T(\epsilon) = 0$ and $T(\omega.n) = g(n)$. Then T is recursive and we have ∂_D partially defined by induction:
  - $\partial_D(\epsilon) = \alpha_0$ is always defined,
  - $\partial_D(\omega.n) = \alpha_{|\omega.n|}$ is defined if $\partial_D(\omega) = \alpha_{|\omega|} = a_1 \to \cdots \to a_n \to \alpha$ and $\alpha_{|\omega.n|} = \alpha_{|\omega|+1} \in a_n$.

  The decoration is infinite since, for all depth d, $\alpha_{d+1} \in \bigcup_{n \leq g(d)} a_n$ for $\alpha_d = a_1 \to \cdots a_{g(d)} \to \alpha'_d$. This contradicts tree-hyperimmunity.
- If D is not tree-hyperimmune, then there is a finitely branching, $\mathbb{N}$-labeled, and recursive tree T and an infinite decoration ∂_D. By König lemma, the sub-tree that constitutes the domain of ∂_D (which is infinite and finitely branching) accepts an infinite branch $(\mu_n)_n$. We denote $\alpha_n := \partial_D(\mu_n)$, so that $\alpha_{n+1} \in a_{T(\mu_{n+1})}$ for $\alpha_n = a_1 \to \cdots \to a_{T(\mu_{n+1})} \to \alpha'$. Since the sequence $(T(\mu_{n+1}))_n$ is majored by the maximal $\mathbb{N}$-label on depth $n+1$ in T, that is recursive, we are contradicting hyperimmunity.

$\square$

[11]König lemma states that any infinite tree that is finitely branching accepts an infinite branch/path.

[12]Trees with nodes labeled by natural numbers.

Remark 2.38. In the following, internal nodes of a quasi-finite Böhm tree are denoted by X, Y... as they are idzntified with the quasi-finite Böhm tree whose root is the node at issue.

We now introduce the notion of the play of a quasi-finite Böhm tree X. The play of X can be seen as the game semantics' play over the infinite arena $* = * \to *$ performed by the execution of X. Formally, it is a (possibly infinite) tree which father-son relationship corresponds to justification pointers. Moreover, players and opponents are playing alternatively, so that nodes at even depth are player nodes and play over applications, and nodes at odd depth are opponent nodes and play over abstractions. We will see that plays over quasi-finite Böhm trees remains finitely branching and recursive trees. Later on, we will try to decorate those plays to contradict tree-hyperimmunity.

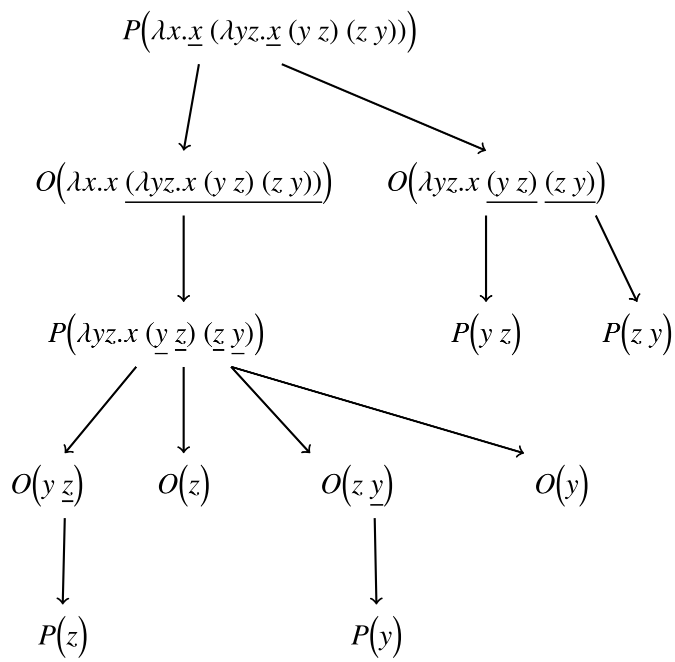
Definition 2.39. Let X be a closed[13]and recursive quasi-finite Böhm tree.
The *play of X* is the recursive and $\mathbb{N}$-labeled tree T whose nodes are of two kinds:

- The nodes at even depth are called *player nodes*. They are denoted $P(Y)$ for some Y over X.
- The nodes at odd depth are called *opponent nodes*. They are denoted $O(Y)$ for some Y over X.

The tree is given by:

- the root is $P(X)$,
- the opponent node $O(\lambda x_1 ... x_m.z\ Y_1 \cdots Y_k)$ has k sons which are the $P(Y_i)$ for $i \leq k$,
- the player node $P(\lambda x_1 ... x_m.z\ Y_1 \cdots Y_k)$ has for sons every $O(Z)$ for Z a node over Y_1, ..., or Y_n whose head variable is one of the $x_1, \ldots, x_m$.

Example 2.40. The tree below is the play over $\lambda x.x\ (\lambda yz.x\ (y\ z)\ (z\ y))$



Proposition 2.41. *Let X be a closed and recursive quasi-finite Böhm tree and T the play over X. For every node Y of X, $P(Y)$ is a node of T. For every node Y of X that is not an Ω, $O(Y)$ is a node of T.*

[13]Can be generalised to non-closed trees by considering plays to be forests of trees.

Proof. By structural induction over the nodes Y of X:

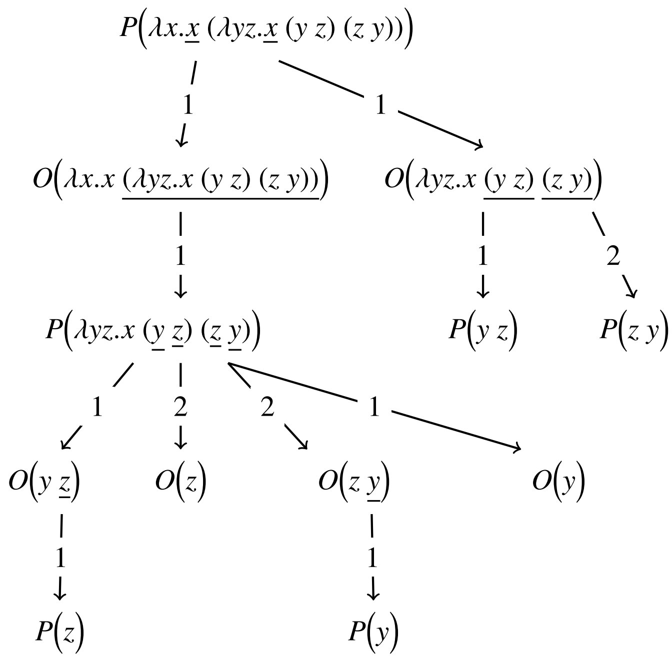
- If Y is a node of X, then either $Y = X$ and $P(X)$ is the root of T, or Y has a father Y' in X. In the last case, $O(Y')$ is a node of T by induction hypothesis and $P(Y)$ is a son of $O(Y')$.
- If $Y' = \lambda x_1...x_m.z\ Y_1 \cdots Y_k$ is a node of X, then by closeness of X, there is an ancestor of Y in X where z is abstracted (potentially $Y = Y'$), *i.e*, $Y = \lambda y_1...y_{m'}.z'\ Y'_1 \cdots Y'_k$ with $z = y_i$. By induction hypothesis, $P(Y)$ is a node of T and $O(Y)$ is its son.

□

Definition 2.42. Let X be a quasi-finite Böhm tree that is recursive and closed.
The *labeled play over* X is the play over X together with the $\mathbb{N}$-labeling ℓ defined as follows:

- the labeling of the root is $\ell(P(X)) = 0$,
- any Y at even depth, $P(Y)$, has for father $O(\lambda x_1...x_m.z\ Y_1 \cdots Y_k)$ with Y one of the Y_i, the $\mathbb{N}$-label $\ell(P(Y))$ is the corresponding index of application i,
- any $Y = \lambda x_1...x_m.z\ Y_1 \cdots Y_k$ at odd depth, $O(Y)$, has for father $P(Y')$ for Y' that is the ancestor of Y in X where z is abstracted (potentially $Y' = Y$), *i.e*, $Y' = \lambda y_1...y_{m'}.z'\ Y'_1 \cdots Y'_k$ with $z = y_i$. The $\mathbb{N}$-label $\ell(O(Y))$ is the corresponding index of abstraction i.

Example 2.43. The tree below is the labeled play over $X = \lambda x.x\ (\lambda yz.x\ (y\ z)\ (z\ y))$. For readability, the label is written in the parent-to-child arrow (we omit $\ell(X) = 0$):



Proposition 2.44. *For any quasi-finite $X \in \boldsymbol{BT_{qf}}$, the labeled play T over X is recursive, finitely branching and $\mathbb{N}$-labeled.*

Proof. The tree T is finitely branching: An opponent node $O(\lambda x_1 \ldots x_n.z\ Y_n \cdots Y_k)$ has exactly k sons which are the $P(Y_i)$ for $i \leq k$. A player node $P(\lambda x_1 \ldots x_n.z\ Y_n \cdots Y_k)$ has one son for each occurrence of its abstracted variables, which results in a finite number by quasi-finiteness of X.

The tree T is recursive: by recursivity and quasi-finiteness of X. □

Our objective is to D-decorate the labeled play of any quasi-finite Böhm tree X such that $[\![X]\!]_{coind} \neq [\![X]\!]_{ind}$. The D-decoration in question will follow a specific patern: we will furnish a path-D-decoration, which is a decoration of the nodes $\{P(Y_n), O(Y_n) \mid n \geq 0\}$ for $(Y_n)_{n\geq 0}$ a path in the Böhm tree of X.

Definition 2.45. Let D be a K-model and X be a quasi-finite Böhm tree where all variables have been named differently.
A path-D-decoration of the labeled play of X is an infinite sequence $(Y_n)_{n\geq 0}$ of nodes of X forming a path (*i.e.*, $Y_0 = X$ and Y_n father of Y_{n+1}) and three infinite sequences $(\alpha^n)_{n\geq 0}, (\beta^n)_{n\geq 0} \in D^{\mathbb{N}}$ and $(a_x)^{x\in \mathrm{FV}(Y_1,Y_2..)}$ such that for each n (Where ℓ is the labeling of Definition 2.42):

$$\begin{aligned}
\beta^n = b^n_1 \to \cdots \to b^n_{\ell(P(Y_{n+1}))} \to \beta' \quad &\Rightarrow \quad \alpha^{n+1} \in b^n_{\ell(P(Y_{n+1}))}.\\
Y_n = \lambda x_1, ..., x_n.y\ X_1 \cdots X_k \quad &\Rightarrow \quad \alpha^n = a_{x_1} \to \cdots a_{x_n} \to \alpha'\\
Y_n = \lambda x_1, ..., x_n.y\ X_1 \cdots X_k \quad &\Rightarrow \quad \beta^n \in a_y
\end{aligned}$$

Proposition 2.46. *Let D be a K-model and X be a quasi-finite Böhm tree.*
A path-D-decoration of the labeled play of X induces a D-decoration of the labeled play of X.

Proof. Let $(Y_n)_{n\geq 0}$, $(\alpha^n)_{n\geq 0}$ and $(\beta^n)_{n\geq 0}$ forming a path-D-decoration of the play of X. Then the partial function ∂_D defined by $\partial_D(P(Y_n)) := \alpha^n$ and $\partial_D(O(Y_n)) := \beta^n$ for all n is a D-decoration:

- the domain of ∂_D is infinite since all Y_n are different (they form a path),
- for any n, the father of $P(Y_{n+1})$ (decorated by α_n) is $O(Y_n)$ which is decorated by β_n and we have by hypothesis
$$\beta^n = b^n_1 \to \cdots \to b^n_{\ell(P(Y_{n+1}))} \to \beta' \quad \Rightarrow \quad \alpha^{n+1} \in b^n_{\ell(P(Y_{n+1}))},$$
- for any n, the father of $O(Y_n)$ is $P(Y_m)$ for some $m \leq n$ such that the head variable y of Y_n is abstracted in the $\ell(O(Y_n))^{th}$ position in Y_m and
$$\begin{aligned}
\alpha^m = a^m_1 \to \cdots \to a^m_{\ell(O(Y_n))} \to \alpha' \quad &\Rightarrow \quad a^m_{\ell(O(Y_n))} = a_y\\
&\Rightarrow \quad \beta^n \in a^m_{\ell(O(Y_n))}.
\end{aligned}$$

□

What follows is a variant of König lemma where we are looking for an infinite path in $\boldsymbol{BT}(X)$ that we can decorate.

Lemma 2.47. Let D be a K-model and $X \in \boldsymbol{BT_{qf}}$ be a quasi-finite Böhm tree. If

$$[\![X]\!]_{coind} \neq [\![X]\!]_{ind},$$

then D is not tree-hyperimmune.

Proof. We can assume that X is closed (otherwise we could have taken $\lambda x_1...x_m.X$)
Let $\alpha \in [\![X]\!]_{coind} - [\![X]\!]_{ind}$.
We define a path-D-decoration of the labeled play of X, breaking the conditions of tree-hyperimmunity by Lemma 2.46. For that we give, inductively, an infinite path $(Y_n)_n$ in X, and three infinite sequences $(\alpha^n)_{n\geq 0}, (\beta^n)_{n\geq 0} \in D^{\mathbb{N}}$ and $(a_x)^{x\in \mathrm{FV}(Y_1,Y_2..)}$ forming the path-D-decoration. Moreover, those are defined such that for all n, $(\vec{a}, \alpha^n) \in [\![Y_n]\!]^{\vec{x}}_{coind} - [\![Y_n]\!]^{\vec{x}}_{ind}$:

- $Y_0 = X$ and $\alpha^0 = \alpha$.
- Assume that we got Y_n. By non emptiness of $[\![Y_n]\!]^{\vec{x}}_{coind}$, we have $Y_n = \lambda x_1...x_m.y\ X_1 \cdots X_k$ with $x_1...x_m$ as free variables:
If we unfold $a_{x_1} \to \cdots a^n_{x_m} \to \alpha' := \alpha^n$, then there exists $\beta^n = b^n_1 \to \cdots b^n_k \to \alpha''_0 \in a_y$ (with

$\alpha'' \geq \alpha'$) such that for all j and all $\gamma \in b^n_j$, we have $(\vec{a}, \gamma) \in [\![X_j]\!]^{\vec{x}}_{coind}$.
In particular there is $j \leq k$ and $\alpha^{n+1} \in b^n_j$ such that $(\vec{a}, \alpha^{n+1}) \in [\![X_j]\!]^{\vec{x}}_{qf} - [\![X_j]\!]^{\vec{x}}_{ind}$.
We set $Y_{n+1} := X_j$ so that

- $\beta^n = b^n_1 \to \cdots b^n_k \to \alpha''_0 \in a_y$ and $\alpha^{n+1} = \gamma \in b^n_j = b^n_{\ell(P(X_j))}$,
- $Y_n = \lambda x_1 ... x_m . x_i\ X_1 \cdots X_k$ and $\alpha^n = a_{x_1} \to \cdots a^n_{x_m} \to \alpha'$,
- $Y_n = \lambda x_1 ... x_m . x_i\ X_1 \cdots X_k$ and $\beta^n \in a_y$.

□

Theorem 2.48. *Any hyperimmune approximable K-model D is also quasi-approximable.*

Proof. We will prove the contrapositive: We assume that D is approximable but not quasi-approximable, then we show that D is not hyperimmune.
Since D is not quasi-approximable, there is a λ-term $M \in \Lambda$ such that $[\![M]\!]^{\vec{x}} \neq [\![\boldsymbol{BT}(M)]\!]^{\vec{x}}_{qf}$.
The approximation property gives that $[\![M]\!]^{\vec{x}} = [\![\boldsymbol{BT}(M)]\!]_{ind} \subset [\![\boldsymbol{BT}(M)]\!]_{qf}$. Thus there is a quasi finite $X \subseteq_{qf} \boldsymbol{BT}(M)$ such that $[\![X]\!]_{coind} \neq [\![X]\!]_{ind}$.
By Lemma 2.47, the K-model D is not tree-hyperimmune and thus not hyperimmune by Lemma 2.37.

□

2.2.2. *Quasi-approximation and extensionality imply full abstraction.*

Theorem 2.49. *Let D be a K-model respecting the quasi-approximation property. Then it is inequationally adequate,* i.e.*, for all M and N such that $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$there is $M \sqsubseteq_{\mathcal{H}^*} N$.*

Proof. D is sensible (diverging terms have empty interpretations). Indeed, for any head-diverging term M, $\boldsymbol{BT}(M) = \Omega$ and thus

$$[\![M]\!]^{\vec{x}} = [\![\boldsymbol{BT}(M)]\!]^{\vec{x}}_{qf} = [\![\Omega]\!]^{\vec{x}}_{qf} = \emptyset.$$

We conclude since sensibility implies inequational adequacy. □

Theorem 2.50. *Let D be a quasi-approximable extensional K-model. D is inequationally complete,* i.e.*, for all M and N; $M \sqsubseteq_{\mathcal{H}^*} N$ implies $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$.*

Proof. Let $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$.
By the quasi-approximation property, there is $W \subseteq_{qf} \boldsymbol{BT}(M)$ such that $(\vec{a}, \alpha) \in [\![W]\!]^{\vec{x}}_{coind}$.
By Proposition 2.13, there are U and V such that $\boldsymbol{BT}(M) \leq_{\eta\infty} U \subseteq V \geq_{\eta\infty} \boldsymbol{BT}(N)$. By applying Lemma 2.21 on $W \subseteq_{qf} \boldsymbol{BT}(M) \leq_{\eta\infty} U$, we get $X \in \boldsymbol{BT_{qf}}$ and by applying it a second time on $X \subseteq_{qf} V \geq_{\eta\infty} \boldsymbol{BT}(N)$ we get Y such that:

$$\begin{array}{ccccccc}
\boldsymbol{BT}(M) & \leq_{\eta\infty} U & \subseteq & V & \geq_{\eta\infty} & \boldsymbol{BT}(N) \\
\cup\!\!|_{qf} & \cup\!\!|_{qf} & & \cup\!\!|_{qf} & & \cup\!\!|_{qf} \\
W & \leq_{\eta\infty} X & = & X & \geq_{\eta\infty} & Y
\end{array}$$

Thus:

$$\begin{array}{rcll}
(\vec{a}, \alpha) \in [\![W]\!]^{\vec{x}}_{coind} & = & [\![X]\!]^{\vec{x}}_{coind} & \text{Lemma 2.35} \\
& = & [\![Y]\!]^{\vec{x}}_{coind} & \text{by Lemma 2.35} \\
& \subseteq & [\![N]\!]^{\vec{x}} & \text{by quasi-approximation}
\end{array}$$

□

2.3. **Full abstraction implies hyperimmunity.**

2.3.1. *The counterexample.*

Suppose that D is approximable but is not hyperimmune. By Definition 1.19 of hyperimmunity, there exists a recursive $g : \mathbb{N} \to \mathbb{N}$ and a sequence $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$ such that

$$\alpha_n = a_{n,1}{\to}\cdots{\to}a_{n,g(n)}{\to}\alpha'_n \qquad\qquad \text{with} \qquad\qquad \alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}.$$

We will use the function g to define a term $\boldsymbol{J}_g$ (Eq. 2.3) such that $(\boldsymbol{J}_g\ \underline{0})$ is observationally equal to the identity in Λ (Lemma 2.53) but can be denotationally distinguished in D (Lemma 2.56). This allows to refute full abstraction:

Theorem 2.51. *If D is approximable but not hyperimmune, then it is not fully abstract for the λ-calculus.*

Basically, $(\boldsymbol{J}_g\ \underline{0})$ is a generalization of the term $\boldsymbol{J}$ used in [13] to prove that the model D^*_∞ (Ex. 1.15) is not fully abstract. The idea is that $\boldsymbol{J}$ is the infinite η-expansion of the identity $\boldsymbol{I}$ where each level of the Böhm tree is η-expanded by one variable. Our term $(\boldsymbol{J}_g\ \underline{0})$ is also an infinite η-expansion of $\boldsymbol{I}$, but now, each level of the Böhm tree is η-expanded by $g(n)$ variables.[14]

Let $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$ be the sequence of closed λ-terms defined by:

$$\boldsymbol{G}_n := \lambda uex_1...x_{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}) \tag{2.1}$$

The recursivity of g implies the recursivity of the sequence $\boldsymbol{G}_n$. Thus, we can use Proposition 1.4: there exists a λ-term $\boldsymbol{G}$ such that:

$$\boldsymbol{G}\ \underline{n} \to^* \boldsymbol{G}_n. \tag{2.2}$$

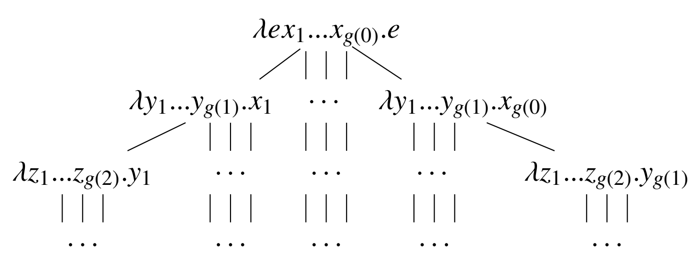
Recall that $\boldsymbol{S}$ denotes the Church successor function and $\boldsymbol{\Theta}$ the Turing fixpoint combinator.
We define:

$$\boldsymbol{J}_g := \boldsymbol{\Theta}\ (\lambda uv.\boldsymbol{G}\ v\ (u\ (\boldsymbol{S}\ v))). \tag{2.3}$$

Then:

$$\boldsymbol{J}_g\ \underline{n} \to^* \boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1}), \tag{2.4}$$

and its Böhm tree can be sketched as



Lemma 2.53 below proves that $\boldsymbol{J}_g\ \underline{0}$ is operationally equivalent to the identity $\boldsymbol{I}$. In fact it is an infinite η-expansion of $\boldsymbol{I}$. But first, we need the following auxiliary lemma.

Lemma 2.52. For any terms $M, N \in \Lambda$ and any fresh z:

$$(M\ z \geq_{\eta\infty} N\ z) \quad\Rightarrow\quad (M \geq_{\eta\infty} N).$$

[14]In the article [9] of the same author, the reader may also find another counterexample based on the same kind of intuitions.

Proof. If M diverges, then so does $(M\ z)$, thus $(N\ z)\Uparrow^{\mathrm{h}}$ and $N\Uparrow^{\mathrm{h}}$, so that $\boldsymbol{BT}(M) = \boldsymbol{BT}(N) = \Omega$. Otherwise we have $M \to_h^* \lambda x_1 \ldots x_n.y\ M_1 \cdots M_k$:

- If $n = 0$, then $M\ z \to_h^* y\ M_1 \cdots M_k\ z$ and $N\ z \to_h^* y\ N_1 \cdots N_k\ z$ with $M_i \geq_{\eta\infty} N_i$, thus $M \geq_{\eta\infty} N$.
- Otherwise, $M\ z \to_h^* \lambda x_2 \ldots x_n.y[z/x_1]\ M_1[z/x_1] \cdots M_k[z/x_1]$ and $N\ z \to_h^* N' \leq_\eta \lambda x_2 \ldots x_n.y[z/x_1]\ N_1 \cdots N_k$ with $M_i[z/x_1] \geq_{\eta\infty} N_i$ for all i. Thus, since z is fresh, $N \to_h^* \lambda x_1.N'[x_1/z] \leq_\eta \lambda x_1 \ldots x_n.y\ N_1[x_1/z] \cdots N_k[x_1/z]$ and $M_i \geq_{\eta\infty} N_i[x_1/z]$, so $M \geq_{\eta\infty} N$.

□

Lemma 2.53. We have $\boldsymbol{J}_g\ \underline{0} \equiv_{\mathcal{H}^*} \boldsymbol{I}$.

Proof. We prove that $(\boldsymbol{J}_g\ \underline{n}\ z) \geq_{\eta\infty} z$ (where z is fresh) for every n, by co-induction and unfolding of $\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z)$:

$$\begin{array}{lr}
\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z) & \\
= \boldsymbol{BT}(\boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1})\ z) & \text{by (2.4)}\\
= \lambda \vec{x}^{g(n)}.z\ \boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n{+}1}\ x_1) \cdots \boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n{+}1}\ x_{g(n)}) & \text{by (2.1)}\\
\geq_{\eta\infty} \lambda \vec{x}^{g(n)}.z\ x_1 \cdots x_{g(n)} & \text{by co-Ind}\\
\geq_\eta z &
\end{array}$$

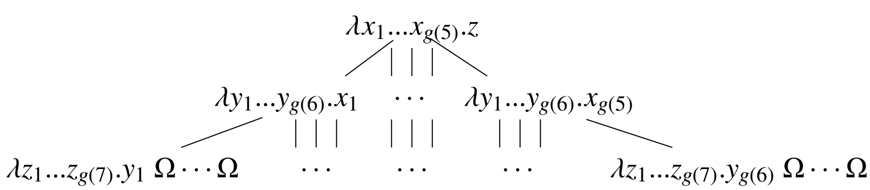
By applying Lemma 2.52, we know that $(\boldsymbol{J}_g\ \underline{n}) \geq_{\eta\infty} \boldsymbol{I}$ and by Corollary 2.15 that $\boldsymbol{J}_g\ \underline{0} \equiv_{\mathcal{H}^*} \boldsymbol{I}$. □

2.3.2. *Denotational separation.*

In this section we show that $\boldsymbol{J}_g\ \underline{0}$ and $\boldsymbol{I}$ are denotationally separated (Lemma 2.56), despite being operationally equivalent.

Let $\boldsymbol{J}_g^{n,k}(z) \in \boldsymbol{BT}_f(\boldsymbol{J}_g\ \underline{n}\ z)$ be the truncation of $\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z)$ at depth k (in particular $\boldsymbol{J}_g^{n,0} = \Omega$).

Example 2.54. For example, $\boldsymbol{J}_g^{5,3}(z)$ is the Böhm tree:



We recall that the sequence $(\alpha_n)_{n\geq 0}$, obtained from the refutation of the hyperimmunity, verifies $\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n$ with $\alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}$.

Lemma 2.55. For all n and k, and for all $a \in \mathcal{A}_f(D)$ such that $\alpha_n \in a$, we have

$$(a, \alpha_n) \notin [\![\boldsymbol{J}_g^{n,k}(z)]\!]_D^z.$$

Proof. By induction on k:

- $(k{=}0)$: since $\boldsymbol{J}_n^{n,0} = \Omega$ then by the approximation property we derive $(a, \alpha_n) \notin [\![\boldsymbol{J}_g^{n,k}(z)]\!]_D^z = \emptyset$.

- $(k+1)$: Remark that $\boldsymbol{J}_g^{n,k+1}(z) = \lambda x_1...x_{g(n)}.z\ (\boldsymbol{J}_g^{n+1,k}(x_1)) \cdots (\boldsymbol{J}_g^{n+1,k}(x_{g(n)}))$ and that for all i, x_i is the only free variable of $\boldsymbol{J}_g^{n+1,k}(x_i)$.
  We unfold $\alpha_n = a_1^n \to \cdots \to a_{g(n)}^n \to \alpha'_n$. Then (a, α_n) belongs to $[\![\boldsymbol{J}_g^{n,k+1}(z)]\!]_D^z$ iff there is $\beta = b_1 \to \cdots \to b_{g(n)} \to \alpha''_n \in a$ (with $\alpha''_n \geq \alpha'_n$) such that for all $i \leq g(n)$ and for all $\gamma \in b_i$, there is $(a_i^n, \gamma) \in [\![\boldsymbol{J}_g^{n+1,k}(x_i)]\!]_D^{x_i}$. The refutation has two cases:
  - For $\beta = \alpha_n$: there is $i \leq g(n)$ such that $\alpha_{n+1} \in b_i = a_i^n$, so that the induction hypothesis gives $(a_i^n, \alpha_{n+1}) \notin [\![\boldsymbol{J}_g^{n+1,k}(x_i)]\!]_D^{x_i}$.
  - For $\beta \neq \alpha_n$, since a is an anti-chain and $\alpha_n \in a$, $\beta \not\geq \alpha_n$. We have seen that $\alpha''_n \geq \alpha'_n$, thus, there is $i \leq g(n)$ such that $b_i \not\leq a_i^n$. In particular, there is $\gamma \in b_i$ such that $\gamma \not\leq \delta$ for any $\delta \in a_i^n$, thus $(a_i^n, \gamma) \notin [\![\boldsymbol{I}\ x_i]\!]_D^{x_i}$. Since $(\boldsymbol{I}\ x_i) \preceq_{\eta\infty} (\boldsymbol{J}_g\ \underline{n+1}\ x_i)$, by applying Lemma 2.28 we obtain $[\![\boldsymbol{I}\ x_i]\!]_D^{x_i} \supseteq [\![\boldsymbol{J}_g\ \underline{n+1}\ x_i]\!]_D^{x_i} \supseteq [\![\boldsymbol{J}_{n+1}^{n+1,k}(x_i)]\!]_D^{x_i}$.

$\square$

Lemma 2.56. The term $\boldsymbol{J}_g\ \underline{n}$ (for any n) and the identity are denotationally separated in D:

$$[\![\boldsymbol{J}_g\ \underline{n}]\!]_D \quad \neq \quad [\![\boldsymbol{I}]\!]_D$$

Proof. Using the approximation property and extensionality, it is sufficient to prove that

$$\{\alpha_0\} \to \alpha_0 \notin \bigcup_k [\![\lambda z.\boldsymbol{J}_g^{n,k}(z)]\!]_D = \bigcup_{U \in \boldsymbol{BT}_f(\boldsymbol{J}_g\ \underline{n})} [\![U]\!]_D = [\![\boldsymbol{J}_g\ \underline{n}]\!]_D,$$

which can be obtained by the application of Lemma 2.55. $\square$

This concludes the proof of the main theorem (Theorem 1.23):

For any extensional approximable K-model D, the following are equivalent:

(1) *D is hyperimmune,*
(2) *D is inequationally fully abstract for* Λ*,*
(3) *D is fully abstract for* Λ*.*

Conclusion

In this paper, we have introduced two very new notions (hyperimmunity and quasi-approximability) on top of two known notions (full abstraction for $\mathcal{H}^*$ and approximability) and a lot of different sub-notions (sensibility, extensionality, theory BT). The relations between these notions may not be clear for the reader, even for classic notions (*e.g.*, few people realize that full abstraction for $\mathcal{H}^*$ does not implies approximability in general).

For such readers, we present, in Figure 5, a graphic summarizing the different properties we have seen in the article. In this figure :

- β stands for being a model (the name refers to the smallest λ-theory β).
- $\mathcal{H}$ stands for the sensible models, *i.e*, those models that equate all diverging terms:
$$M, N \Uparrow^{\mathrm{h}} \quad \Rightarrow \quad [\![M]\!] = [\![N]\!].$$
- $\beta\eta$ stands for extensional models, *i.e*, those models preserving η-equivalence:
$$[\![\lambda x.x]\!] = [\![\lambda xy.x\ y]\!].$$
- $\boldsymbol{BT}$ stands for models that respect Böhm trees:
$$\forall M, N, \quad \boldsymbol{BT}(M) = \boldsymbol{BT}(N) \quad \Rightarrow \quad [\![M]\!] = [\![N]\!].$$

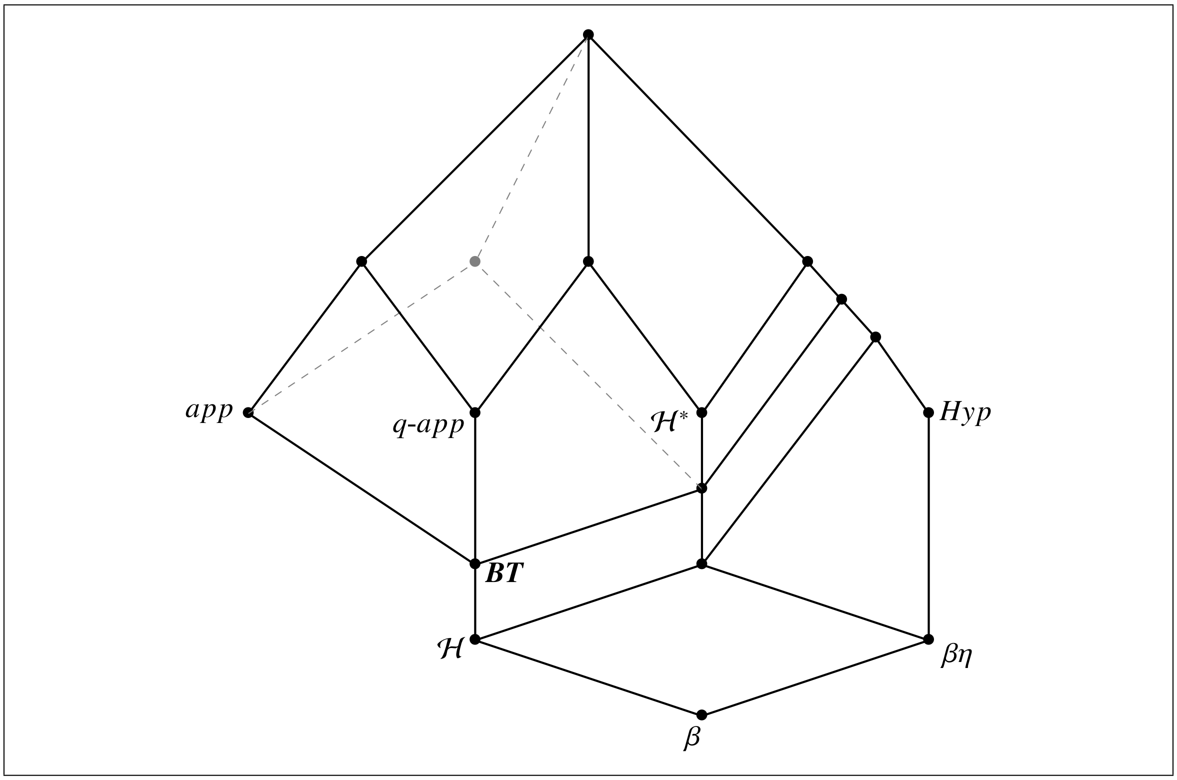


FIGURE 5. Lattice of the properties considered in this paper.

- $\mathcal{H}^*$ stands for models that are fully abstract for $\mathcal{H}^*$:
$$\forall M, N, \quad M \equiv_{\mathcal{H}^*} N \quad \Leftrightarrow \quad [\![M]\!] = [\![N]\!].$$
- *app* stands for models that are approximable:
$$\forall M, \quad [\![M]\!] = [\![\boldsymbol{BT}(M)]\!]_{ind}.$$
- *q-app* stands for models that are quasi-approximable:
$$\forall M, \quad [\![M]\!] = [\![\boldsymbol{BT}(M)]\!]_{qf}.$$
- *Hyp* stands for models that are hyperimmune.
- The other nodes are simply defined as sups and do not have names.

This graphic is a lattice of properties that a K-model can satisfy, with binary sups corresponding to the conjunction of the properties (modulo logical equivalence).[15] In particular, one can see that quasi-approximation together with extensionality implies the full abstraction for $\mathcal{H}^*$. Moreover, for any among our four main properties (*i.e.*, *app*, $\mathcal{H}^*$, *q-app* and Hyp), having any two non-adjacent properties (*app*/$\mathcal{H}^*$, *app*/*Hyp* or *q-app*/Hyp) is sufficient to get the two others.

Notice that in the article we are claiming that *app* and *q-app* implies Hyperimmunity, but this was in presence of extensionality. One can then check that the sup of *app*, *q-app* and $\beta\eta$ is indeed the top of our lattice.

Notice also that we placed hyperimmunity above extensionality. This is because we use extensionality in order to define hyperimmunity. A careful reader may probably be able to extend

[15]Notice that two points in the graphic may well be logically equivalent.

naturally hyperimmunity to a non-extensional setting, but several of the relations of Figure 5 may break with this generalization.

Finally, we conjecture that all these relations are strict in the fully general case (extended to models that are not K-models). This is proved for most already existing relations but not for the relations between $\boldsymbol{BT}$, app and $\mathcal{H}^*$.

Approximability is not *a propri* implied by $\boldsymbol{BT}$ or even by $\mathcal{H}^*$ but no counter-examples have been presented yet. This is a difficult question related to the characterization of sensibility. In fact it is actually difficult to get an idea of what non-approximable models lies above $\mathcal{H}$. Indeed, the most efficient methods we know for proving sensibility are realisability methods that are intrinsically linked with approximability [7]. Notice that the only result on this direction was from Kerth that created a continuum of sensible models of (disjoint) theories below $\boldsymbol{BT}$ [23]. In this paper we simply avoid the difficulty by only considering approximable K-models.

This was the first attempt at studying a λ-theory by characterising its fully abstracting models (among a relatively large class). This opens a lot of new research directions such as generalisations for larger classes of models, for other languages or for other λ-theories. The latter has actually been explored by the author in a collaborative work on Morris's extensional equivalence (the observational equivalence for weak reduction) [11]. This work is bounded to relational models which are morally extensional extensions[16] of approximable K-models [5]. There, we show that the full abstraction for Morris's equivalence corresponds to satisfy the λ-Konig property. The λ-Konig property is a sort of dual of hyperimmunity: rather than forbidding all infinite non-hyperimmune chains, it requires the presence of a dense set of such non-hyperimmune chains.

Acknowledgements

I wish to thank Antonio Bucciarelli, Michele Pagani, Antonino Salibra and the anonymous reviewers for their proof-checking and helping in clarifying many issues. I must especially thank Michele for his time spent in the supervision of the redaction of this paper. Finally I acknowledge Thomas Ehrhard and Giulio Manzonetto for their advices and various discussions.

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